15-819/18-879: Logical Analysis of Hybrid Systems

10: Hybrid Systems Image Computation

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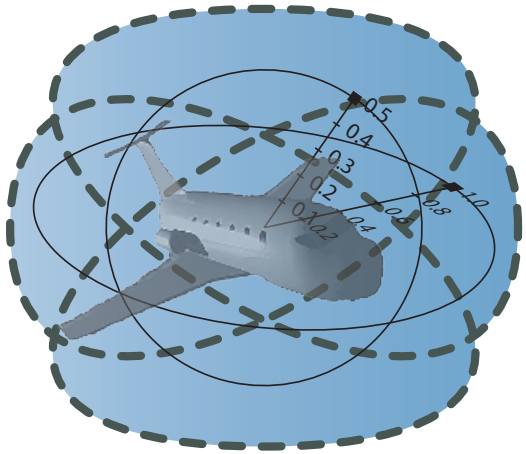

Outline

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Model Checking in a Nutshell

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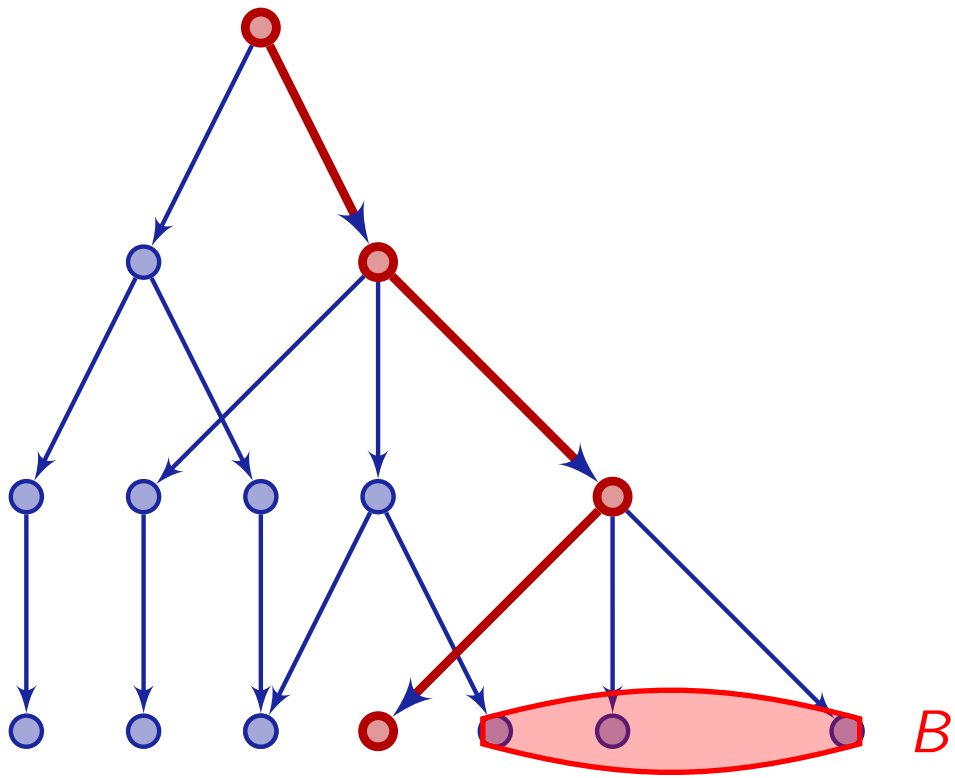
Given initial states $Q_0 \subseteq Q$ and bad states $B \subseteq Q$ for a transition system, check whether there is a trace from some $q_0 \in Q_0$ to some $q_b \in B$.



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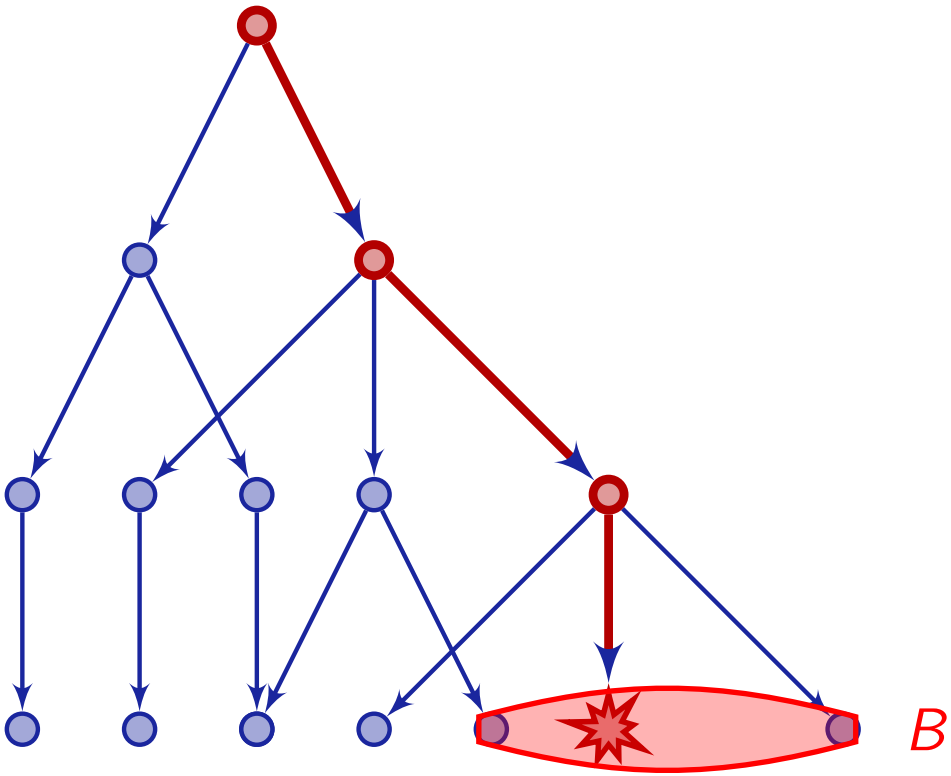

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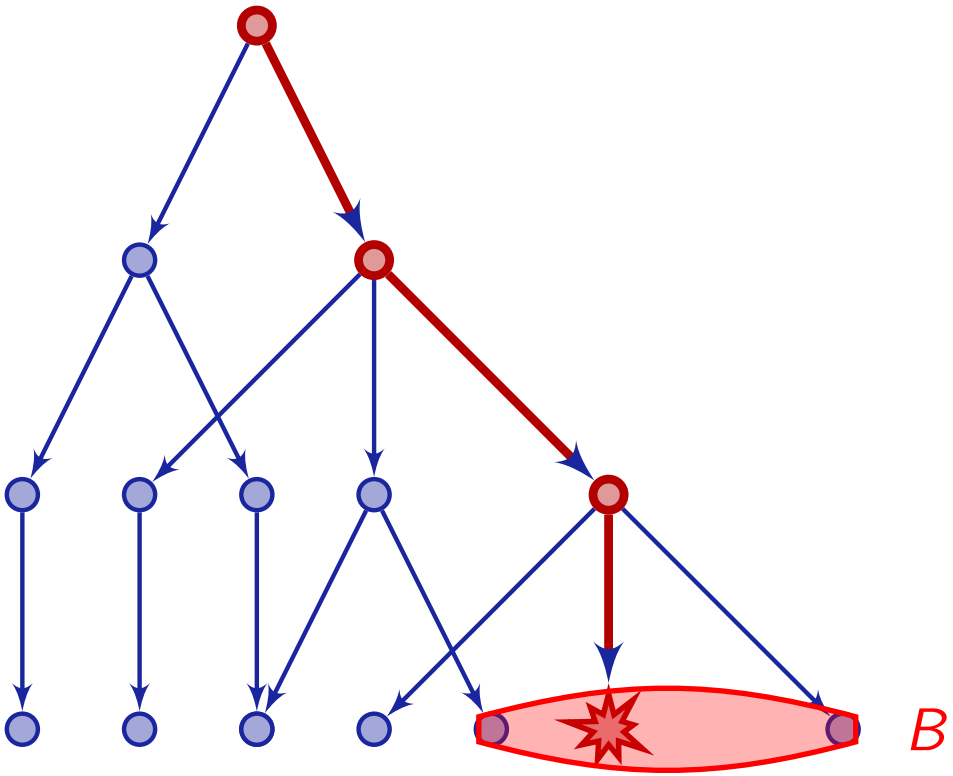

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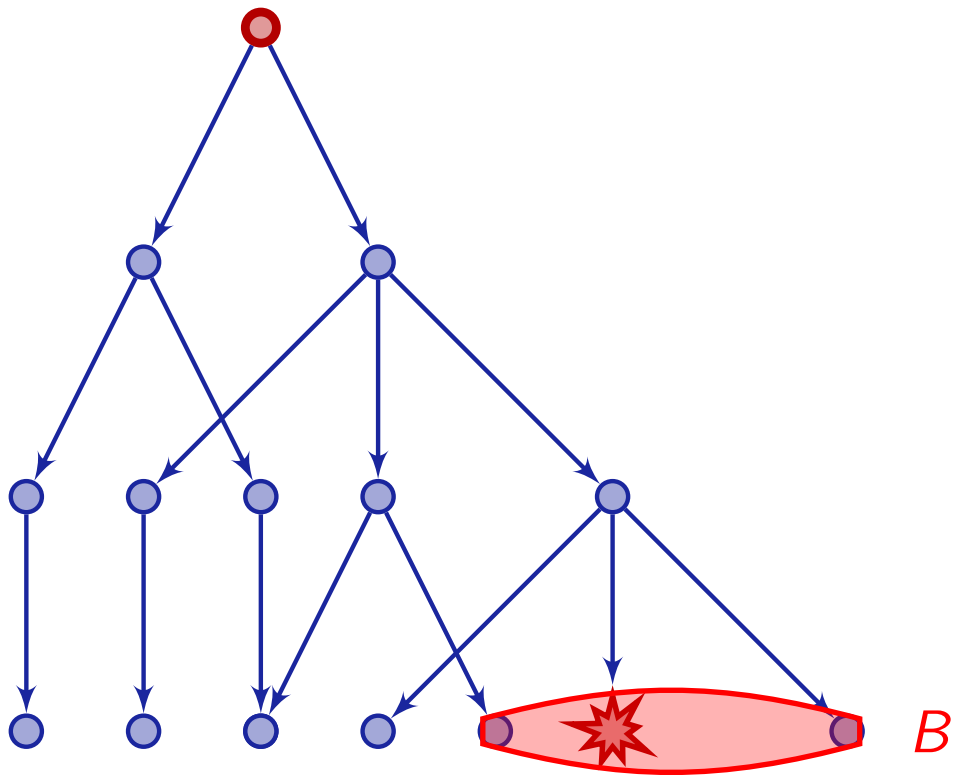

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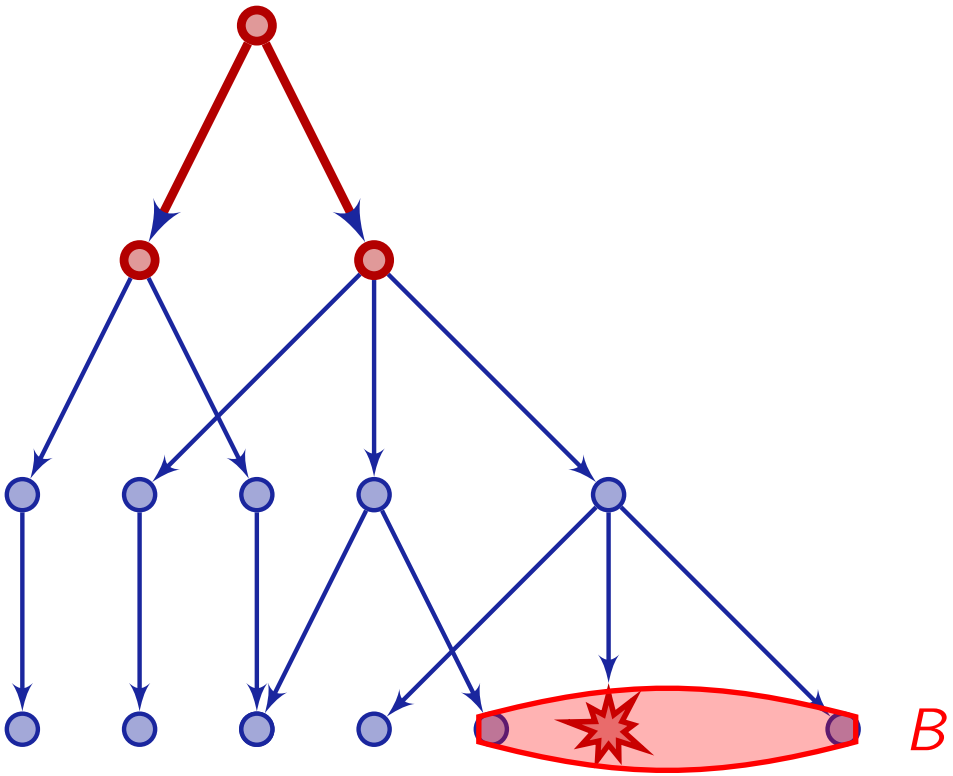

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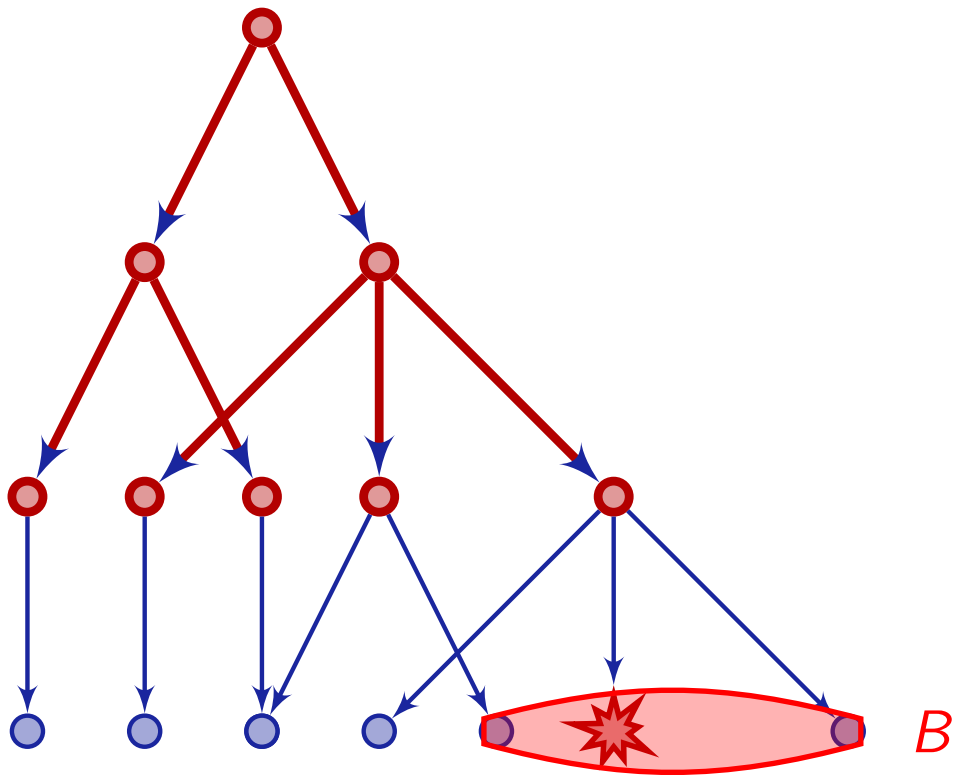

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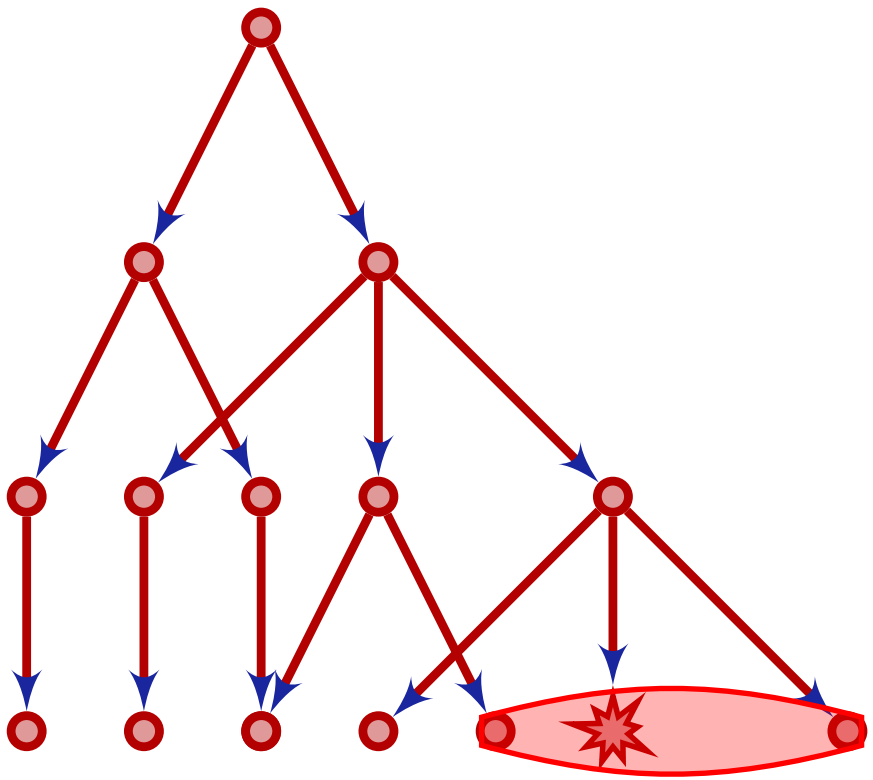

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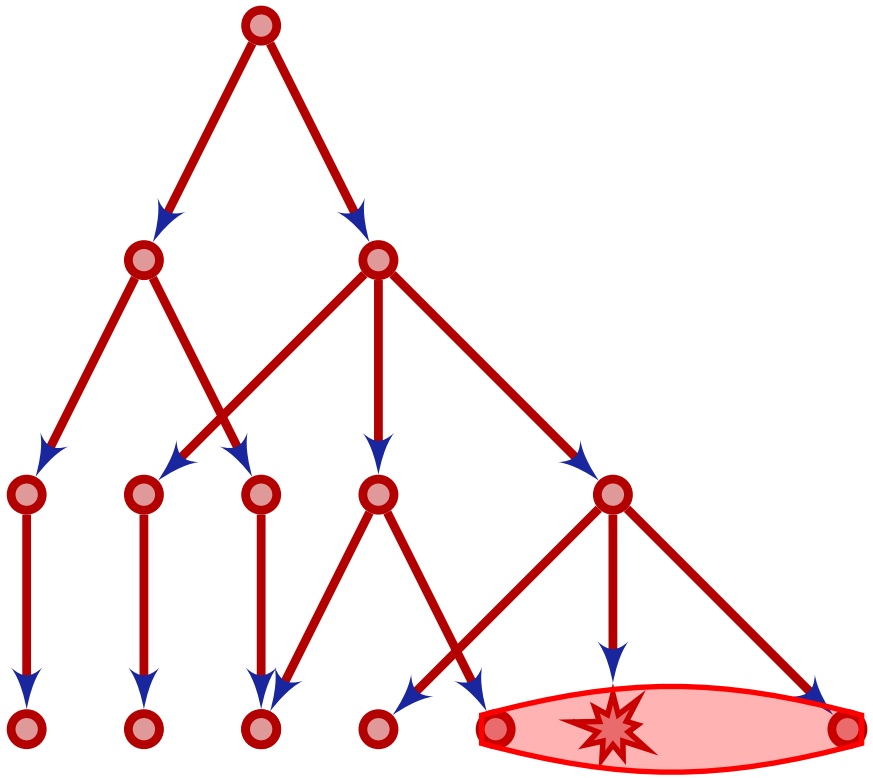

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Can we use this for hybrid systems?

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For finite-state systems, this naïve MC algorithm gives a (slow) decision procedure.

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Hybrid systems have uncountable state spaces

(Uncountably) infinite state spaces require extra care

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- Simulation-style $Post_A(Y)$ simple but more problems than solutions.

HS BMC: Numerical Finite Discrete-Time

Discrete-time finite-state numerical bounded model checking

Approximate $Post_A^*(Y)$ for finite $Y \subseteq Q \times \mathbb{Q}^n$ on a grid $A := E \cup \{\Delta\}$

1. $Post_e(Y) := \{(q^+, x^+) \ : \ \exists (q,x) \in Y, x \models guard_e, (x, x^+) \models reset_e\}$ for edge $e \in E$ going from q to q^+
2. $Post_\Delta(Y) := \{(q, \varphi_q(\Delta, x)) \ : \ (q,x) \in Y, \varphi_q(\Delta, x) \models inv_q\}$ hoping that $\varphi_q(t,x) \models inv_q$ for all $0 \leq t \leq \Delta$ assuming that $\varphi_q(t,x)$ is computable
3. $Y := Post_A(Y) := Post_\Delta(Y) \cup \bigcup_{e \in E} Post_e(Y)$
4. Repeat until some finite number of steps (bounded model checking)

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- "$z \in Post_{p|_D}(Y)$" for evolution domain $D := inv_q$ is definable as:

$$\exists x \, \exists t \geq 0 \, (F_Y(x) \wedge \forall 0 \leq s \leq t \, F_D(s, p(s,x)) \wedge z = p(t,x))$$

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Hybrid Systems Model Checking

How to compute $Post_A(Y)$? Then $Post_A^*(Y)$ won't be long?

- Consider the case where $Y \subseteq Q$ is (infinite) set.
- How to represent such sets of states computationally?
- Consider polyhedra, ellipsoid, zonotope, semialgebraic set Y
- Let Y be defined by $\mathrm{FOL}_{\mathbb{R}}$ formula F_Y, i.e., $Y = \{a \in \mathbb{R}^n \ : \ a \models F_Y\}$ and evolution domain D by formula F_D
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Continuous-time symbolic bounded model checking

Compute $Post_A^N(F_Y)$ for symbolic representation $F_Y : Q \to \mathrm{FOL}_{\mathbb{R}}$

1. $Post_{\mathbb{R}_{\geq 0}}(F_Y) := \{(q, G_q(F_Y(q))) \ : \ q \in Q\}$

   $G_q(L) := QE(\exists z\, \exists t{\geq}0\, (L(z) \wedge \forall 0{\leq}s{\leq}t\, F_D(s, \varphi_q(s,z)) \wedge x = \varphi_q(t,z)))$

   assuming that $\varphi_q(t,z)$ is polynomial and F_D represents inv_q.
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HS BMC: Symbolic Continuous-Time

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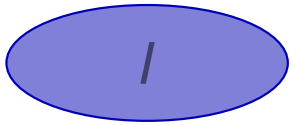
× Not sound (no problem found does not mean safe)

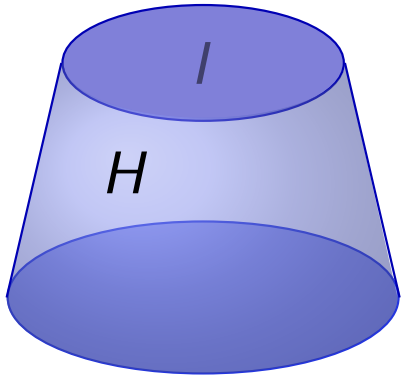
What analysis is doable at all?

Image Computation in Hybrid Systems



- Analyse image computation problem in hybrid systems
- Approximation refinement techniques and their limits
- Numerical versus symbolic algorithms
  $1.421 \in \mathbb{Q}$ versus $x^2 + 2xy$ term computations

Image Computation in Hybrid Systems



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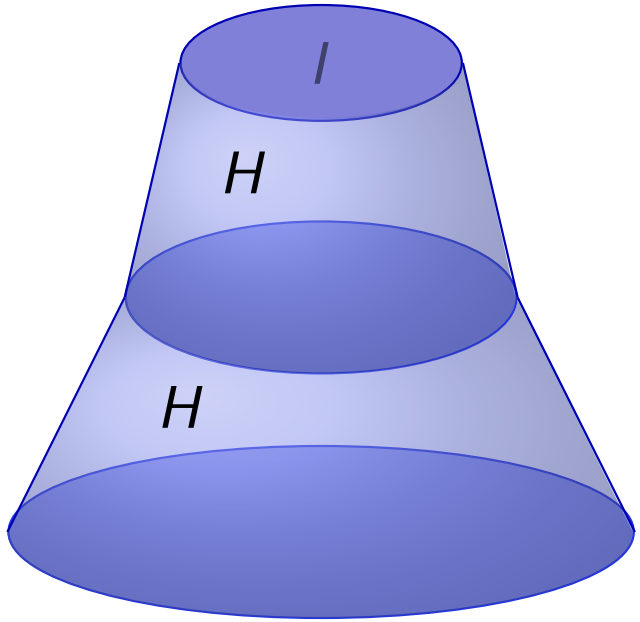

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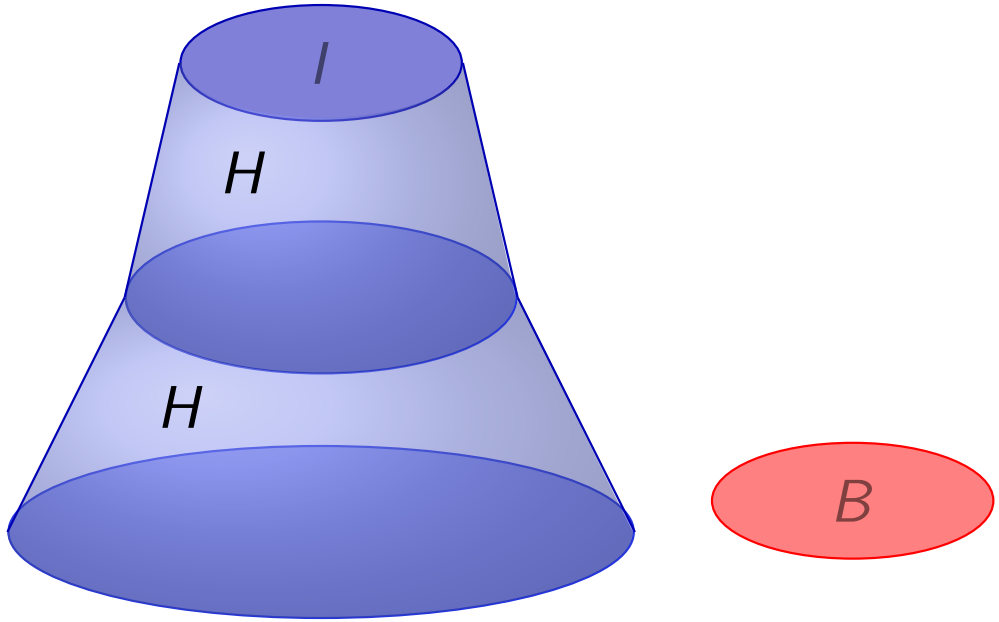

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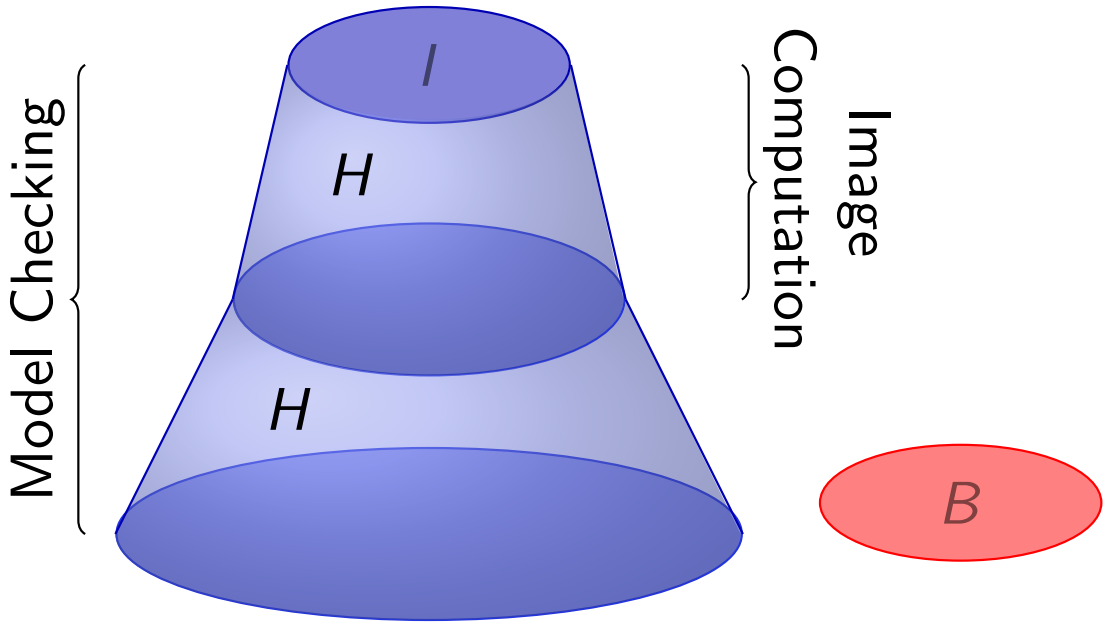

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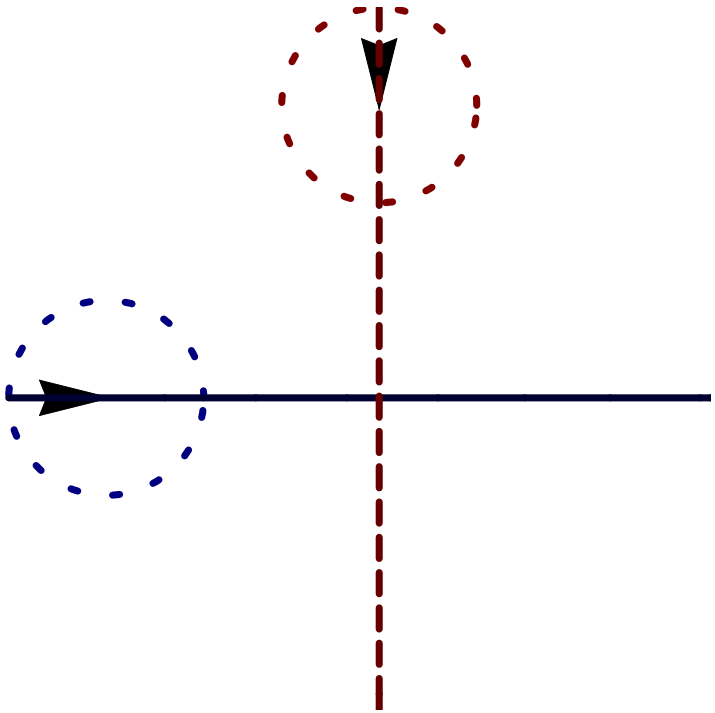
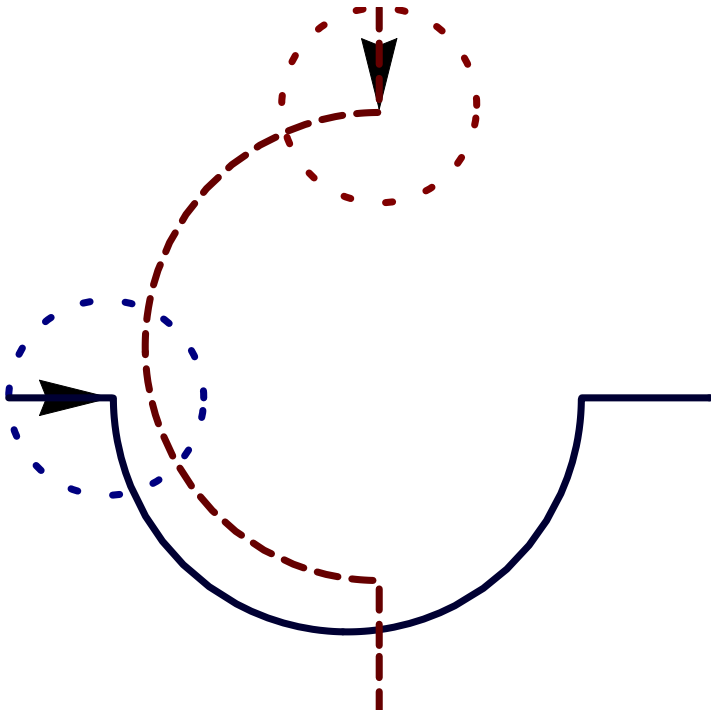
Air Traffic Management



Air Traffic Management



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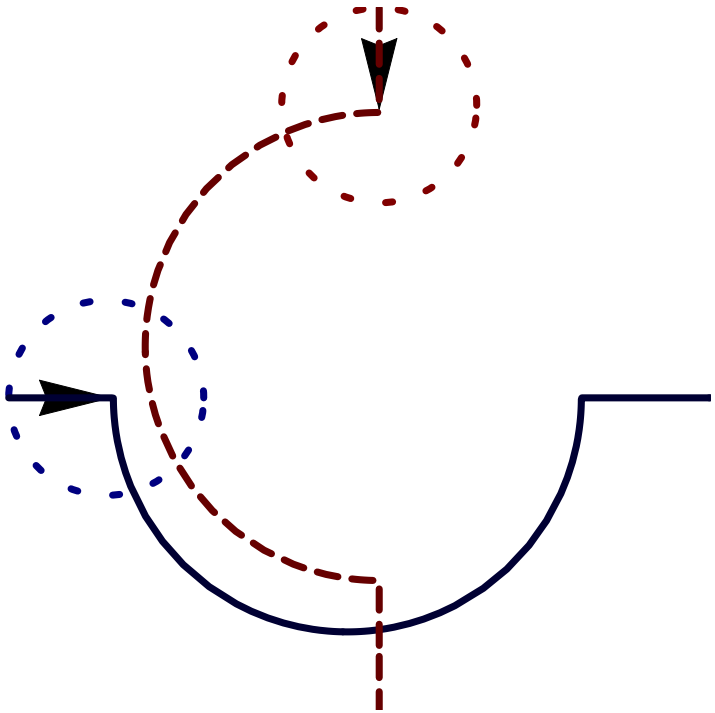
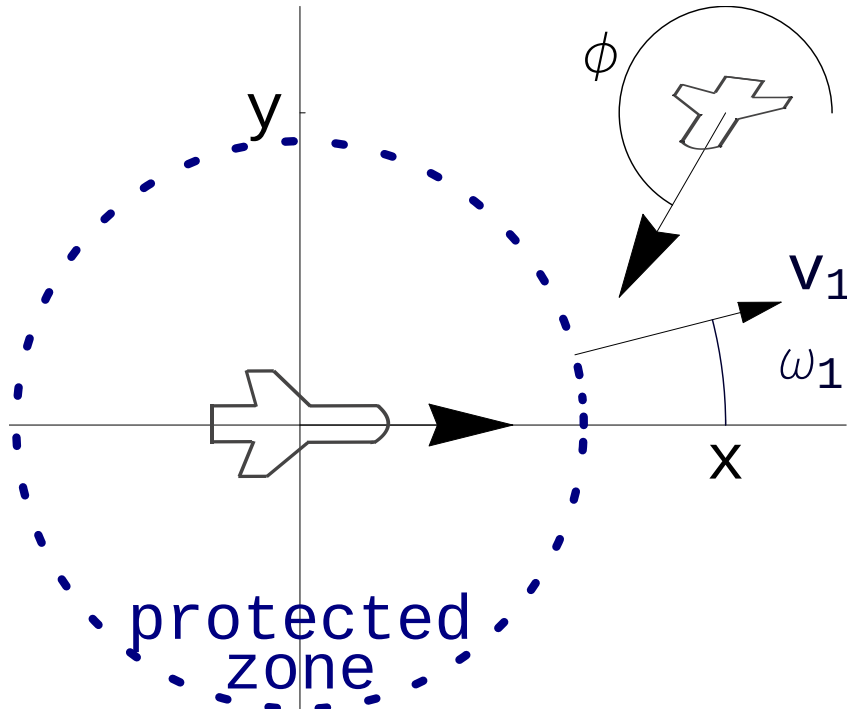
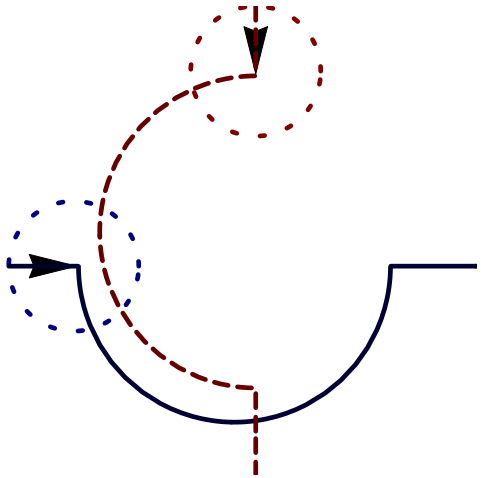
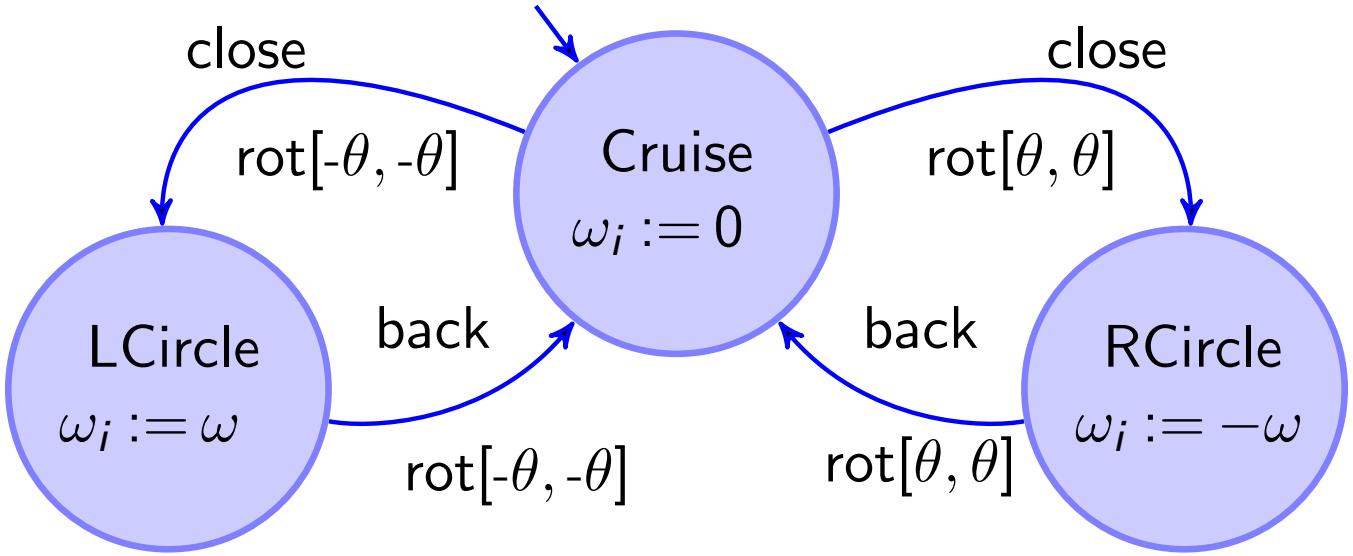




ATM: Roundabout Maneuver Automaton



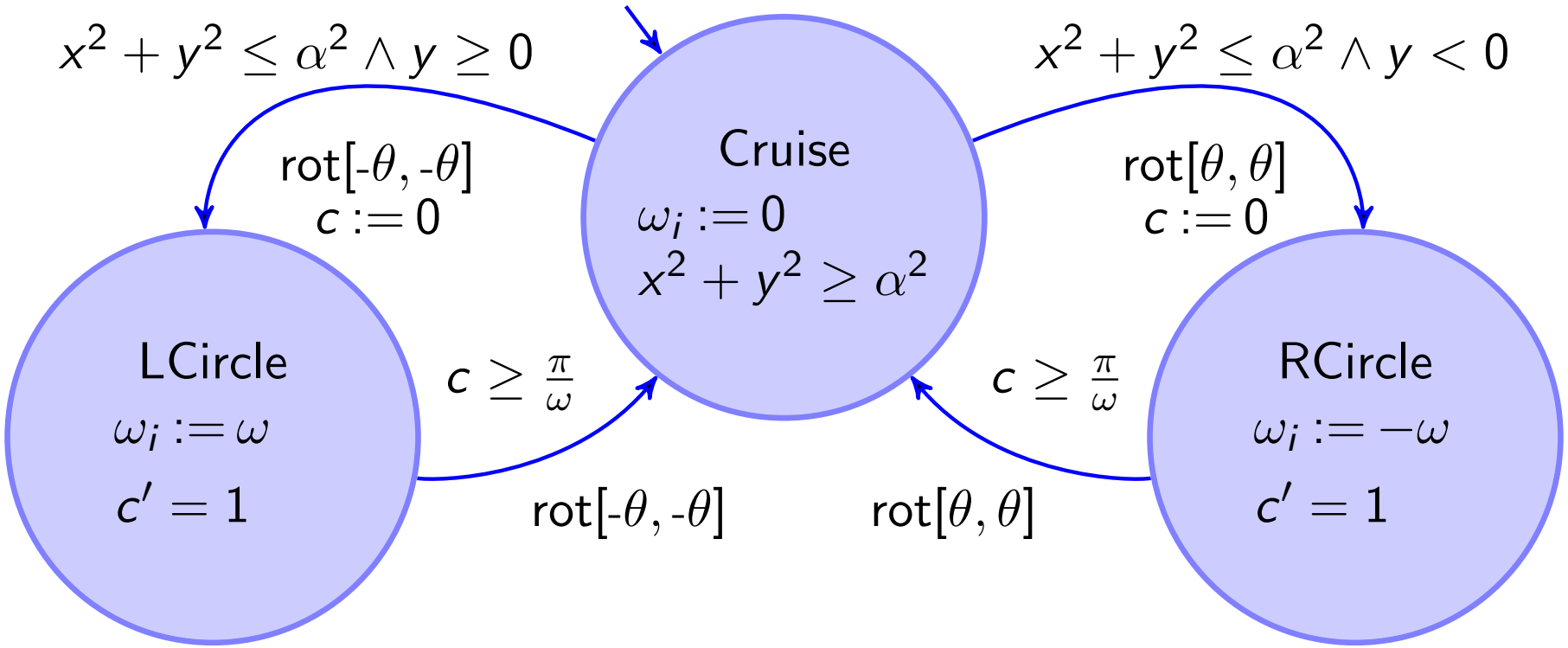
$$\begin{bmatrix} x' &= -v_1 &+v_2 \cos\phi &+\omega_1 y \\ y' &= &v_2 \sin\phi &-\omega_1 x \\ \phi' &= &\omega_2 &-\omega_1 \end{bmatrix}$$



▸ Details

ATM: Roundabout Maneuver Automaton

$$\begin{bmatrix} x' = -v_1 + v_2 \cos\phi + \omega_1 y \\ y' = v_2 \sin\phi - \omega_1 x \\ \phi' = \omega_2 - \omega_1 \end{bmatrix}$$



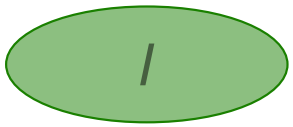
Return

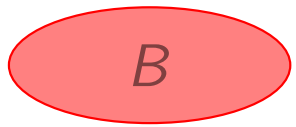
Outline

AMC: Approximation Refinement Model Checking

AMC(B reachable from I in H):

1. $A := \text{approx}(H)$ uniformly
2. blur by uniform approximation error $+\epsilon$
3. check(B reachable from I in $A + \epsilon$)
4. B not reachable $\Rightarrow$ H safe





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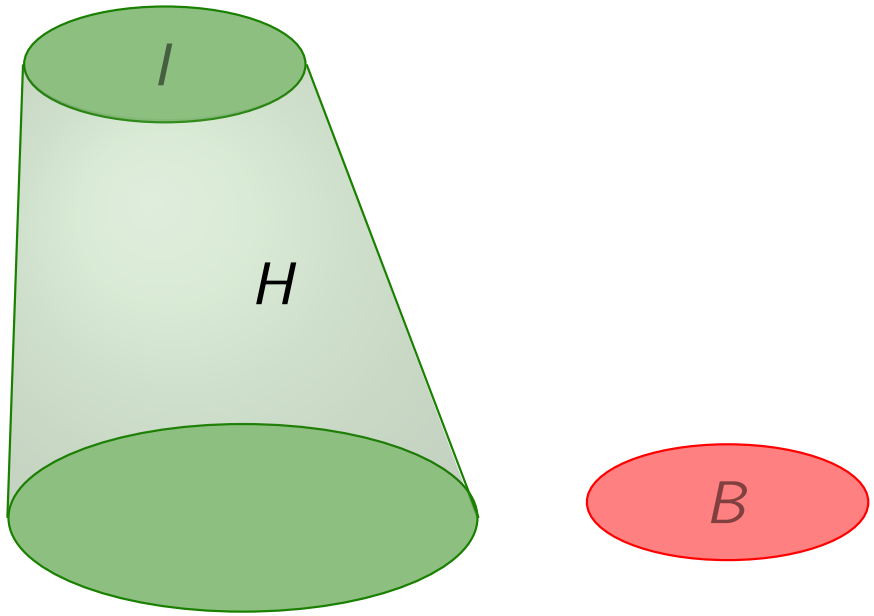

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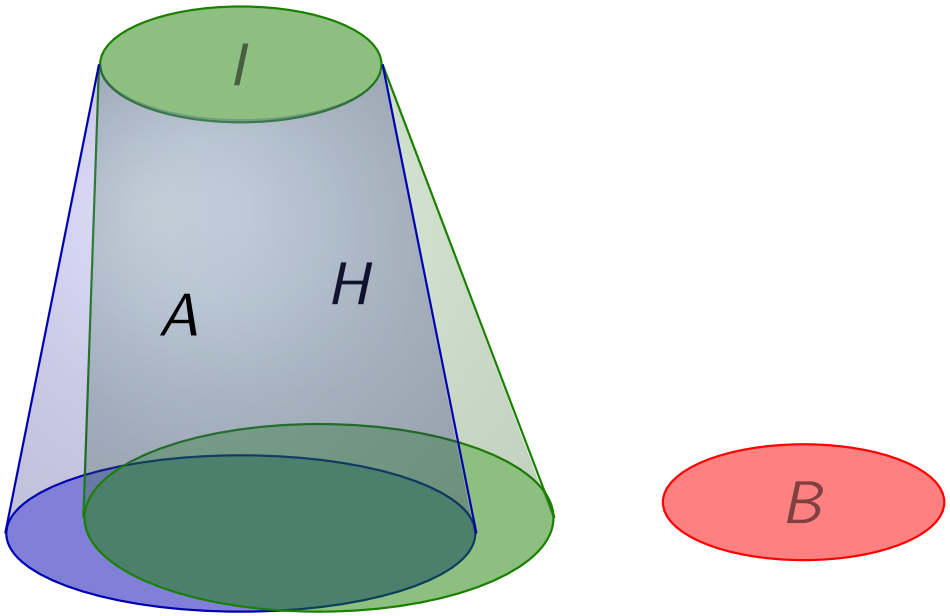

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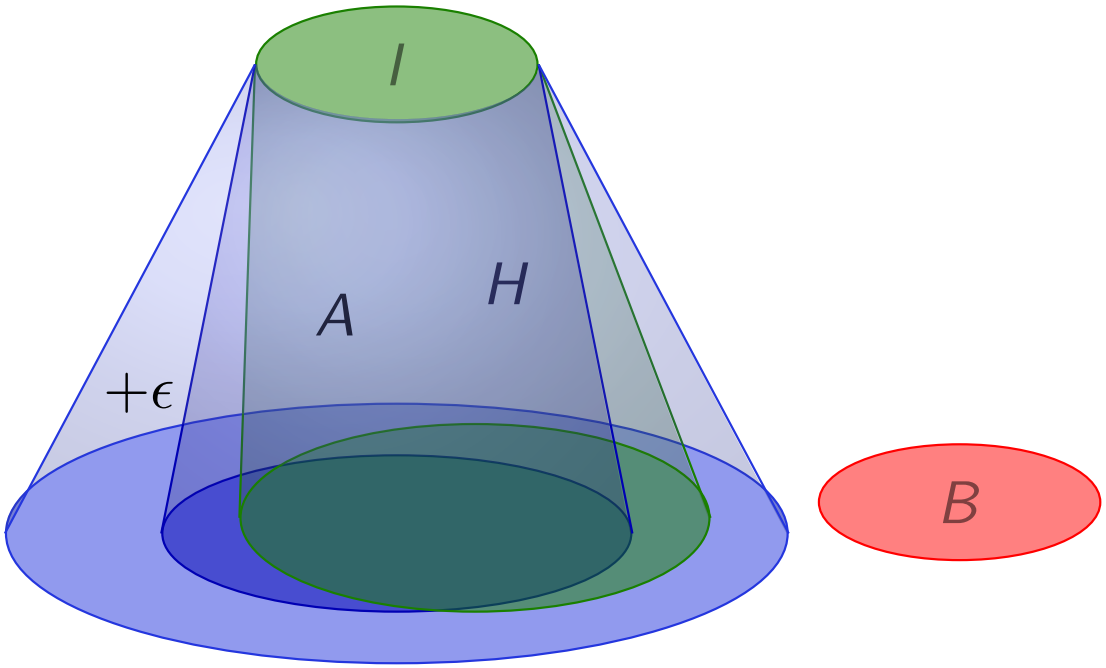

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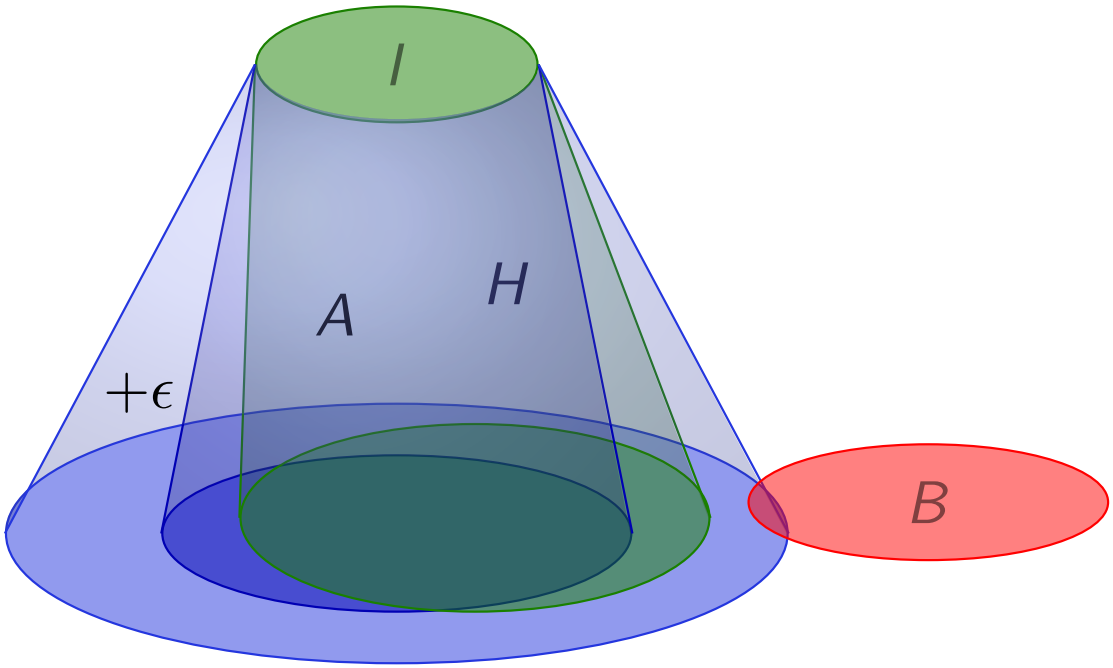

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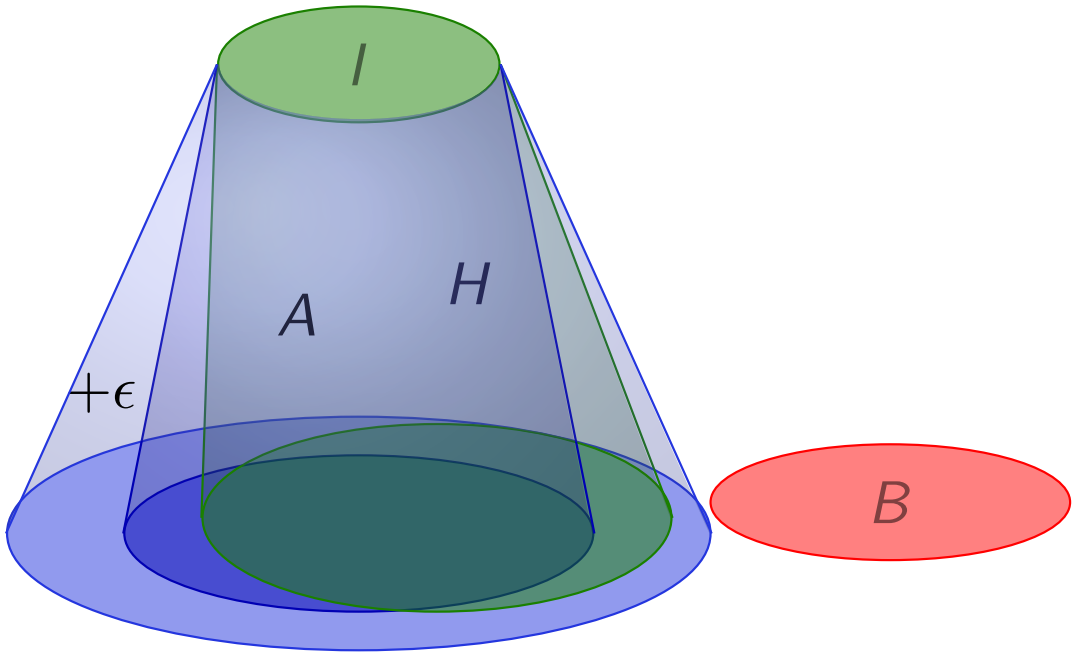

AMC: Exact Image Computation

AMC(B reachable from I in H):

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Proposition

check and blur can be implemented for

- *I and B semialgebraic (propositional combinations of $p \geq 0$)*
- *A with polynomial flows over $\mathbb{R}$*
- *$+$Piecewise definitions*
- *$+$Rational extensions (e.g. multivariate rational splines)*

AMC: Image Approximation

AMC(B reachable from I in H):

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Proposition

approx exists for all uniform errors $\epsilon > 0$ when

- *using polynomials to build A*
- *Flows $\varphi \in C(D, \mathbb{R}^n)$ of H*
- *$D \subset \mathbb{R} \times \mathbb{R}^n$ compact closure of an open set*

Approximation can solve problems without effective exact solution

Weierstraßian Flow Approximation

Proposition

approx exists for all uniform errors $\varepsilon > 0$*:*

- $\varphi \in C(D, \mathbb{R}^n)$ *on compact closure* $D \subset \mathbb{R} \times \mathbb{R}^n$ *of an open set*

$\Rightarrow \quad \forall \varepsilon > 0 \;\; \exists p \in \mathbb{R}[t, x_1, ..., x_n]^n \, \forall Y \subseteq \mathbb{R}^n$

$$Post_{\varphi|_D}(Y) \;\subseteq\; \mathcal{U}_\varepsilon(Post_{p|_D}(Y))$$

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$$Post_{\varphi|_D}(Y) \;\subseteq\; \mathcal{U}_\varepsilon(Post_{p|_D}(Y))$$

Where $\mathcal{U}_\varepsilon(Y)$ is the ε ball around set Y:

$$\mathcal{U}_\varepsilon(Y) := \{x \;:\; \|x - y\| < \varepsilon \text{ for some } y \in Y\}$$

Weierstraßian Flow Approximation

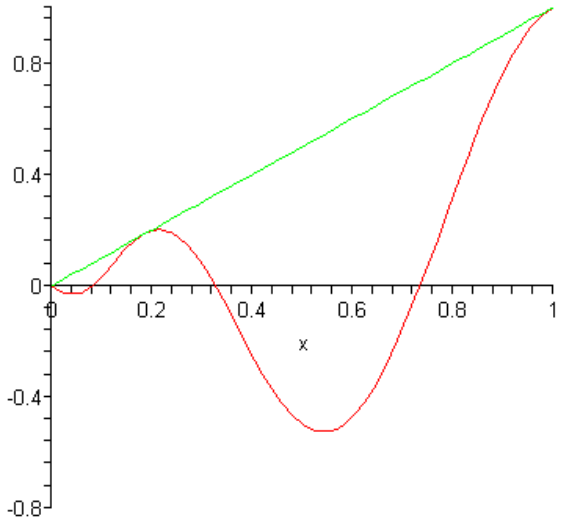
Theorem (Stone-Weierstraß Approximation)

Polynomials uniformly approximate cont. functions on compact domains:

- $\varphi \in C(D, \mathbb{R}^n)$ *on compact domain* $D \subset \mathbb{R} \times \mathbb{R}^n$

$\Rightarrow \quad \forall \varepsilon > 0 \;\; \exists p \in \mathbb{R}[t, x_1, ..., x_n]^n \, \forall (t, x) \in D$

$$\|\varphi(t; x) - p(t, x)\| < \varepsilon$$



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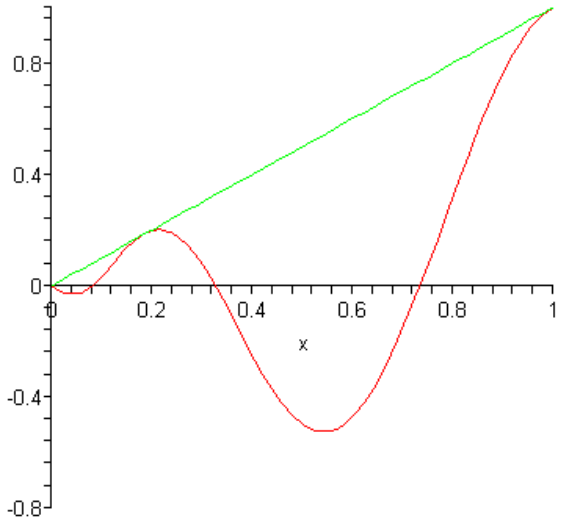

Existence of solutions may be computationally insufficient

Exact Image Computation: Polynomials

Proposition

check and *blur* can be implemented for

- I, D, B definable in $\mathrm{FOL}_{\mathbb{R}}$, *i.e., semialgebraic*
- A *with polynomial flows over* $\mathbb{R}$

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Proof.

Inductive consequence of $\mathcal{U}_{\varepsilon}(Post_{p|_D}(Y))$ being definable in $\mathrm{FOL}_{\mathbb{R}}$, thus being decidable: Let Y, D be defined by $\mathrm{FOL}_{\mathbb{R}}$ formulas F_Y, F_D.

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2. "$z \in \mathcal{U}_\varepsilon(Y)$" is definable in $\mathrm{FOL}_{\mathbb{R}}$, thus decidable:

$$\exists y\, (F_Y y \wedge \sum_{i=1}^{n} (y_i - z_i)^2 < \varepsilon^2)$$

Exact Image Computation: Piecewise Polynomials

Proposition

check and *blur* can be implemented for

- I, D, B definable in $\mathrm{FOL}_{\mathbb{R}}$, i.e., semialgebraic
- A with *piecewise* polynomial flows over $\mathbb{R}$

Exact Image Computation: Piecewise Polynomials

Proposition

check and *blur* can be implemented for

- I, D, B definable in $\mathrm{FOL}_{\mathbb{R}}$, i.e., semialgebraic
- A with *piecewise* polynomial flows over $\mathbb{R}$

Proof.

$s : D \to \mathbb{R}$ consists of polynomial pieces $p_i : D_i \to \mathbb{R}$ for disjoint definable D_i with $D = D_1 \cup \ldots \cup D_n$. Then, we define $\mathcal{U}_\varepsilon(Post_{s|_D}(Y))$:

Exact Image Computation: Piecewise Polynomials

Proposition

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$s : D \to \mathbb{R}$ consists of polynomial pieces $p_i : D_i \to \mathbb{R}$ for disjoint definable D_i with $D = D_1 \cup \ldots \cup D_n$. Then, we define $\mathcal{U}_\varepsilon(Post_{s|_D}(Y))$:

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Exact Image Computation: Piecewise Polynomials

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check and *blur* can be implemented for

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$$Post_{s|_D}(Y) = \bigcup_{i=1}^{n} Post_{p_i|_{D_i}}(Y) \quad \text{and} \quad \mathcal{U}_\varepsilon(X \cup Y) = \mathcal{U}_\varepsilon(X) \cup \mathcal{U}_\varepsilon(Y)$$

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$$\begin{aligned} p(x)/q(x) = 0 &\equiv p(x) = 0 \wedge q(x) \neq 0 \\ p(x)/q(x) > 0 &\equiv (p(x) > 0 \wedge q(x) > 0) \vee (p(x) < 0 \wedge q(x) < 0) \end{aligned}$$

□

Logical foundation for effective image computation operations

Outline

Bounded Flow Approximation

Proposition (Effective Weierstraß approximation)

- *Flows* $\varphi \in C^1(D, \mathbb{R}^n)$ *arbitrarily effective,* D *effective*
- *Bounds* $b := \max_{x \in D} \|\varphi'(x)\|$

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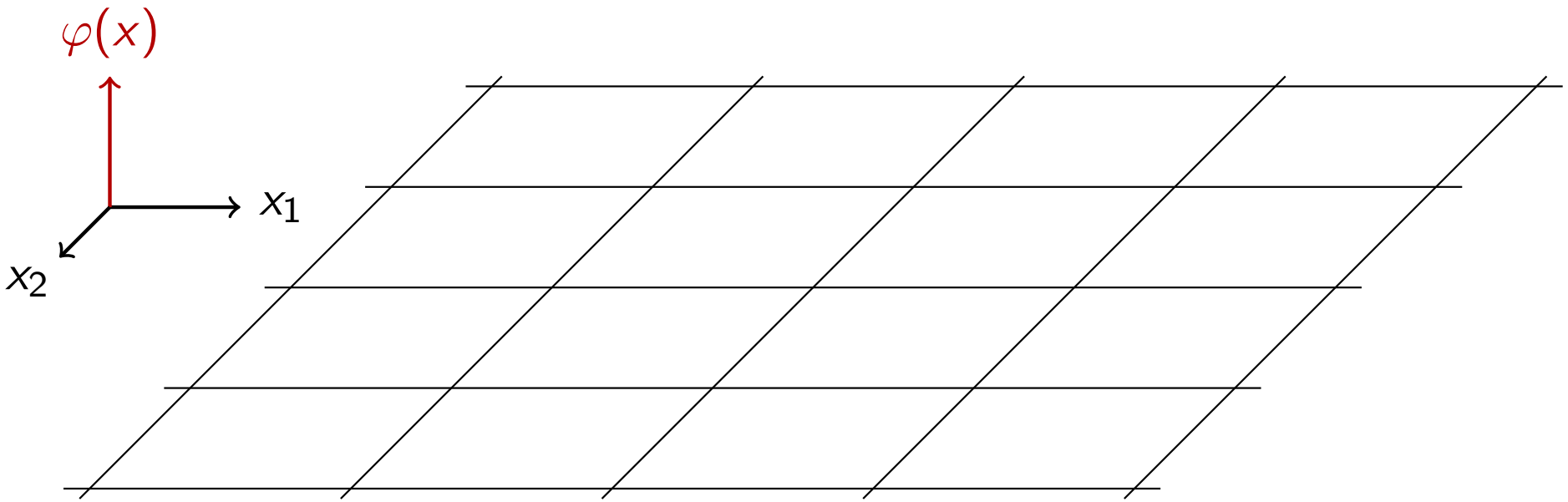

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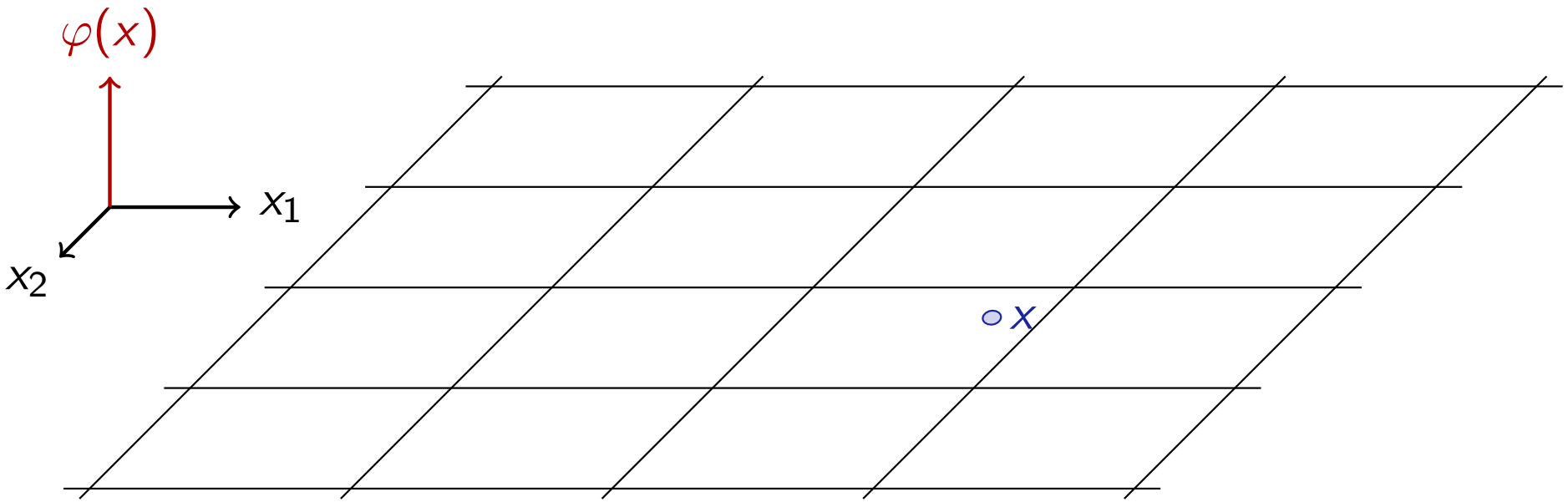

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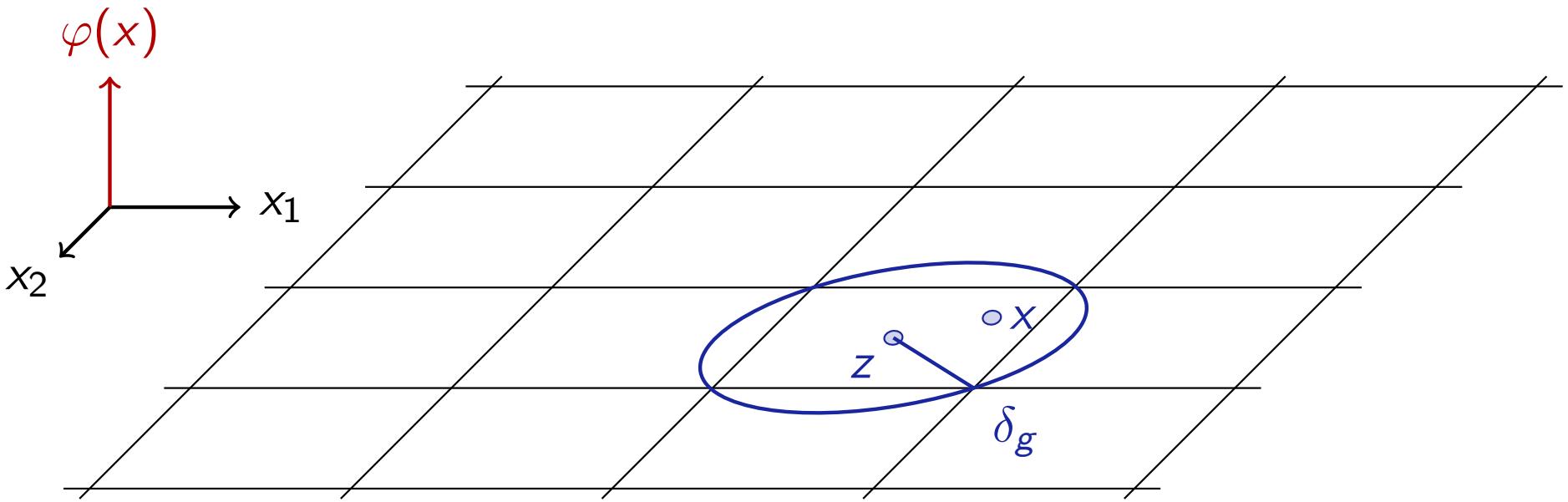

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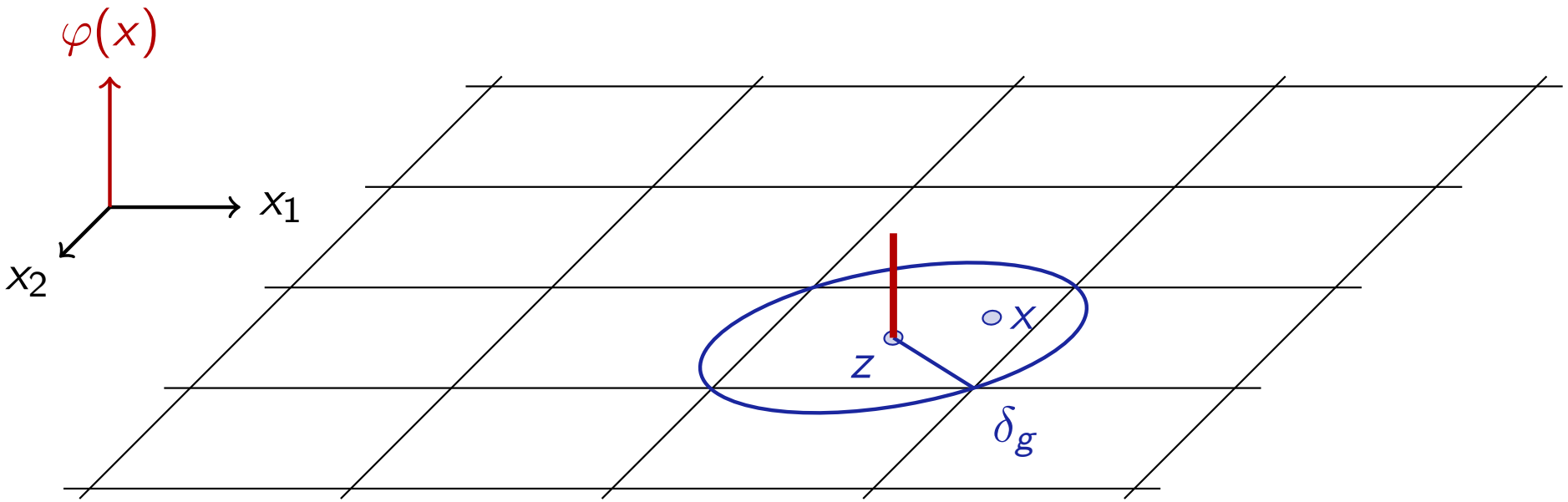

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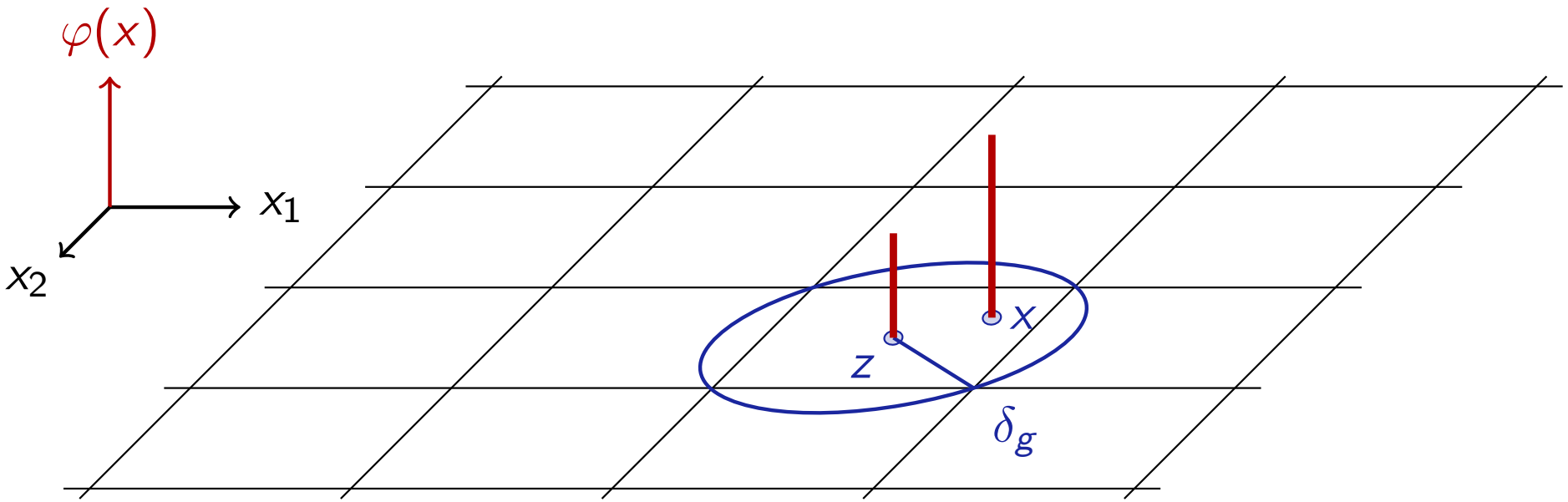

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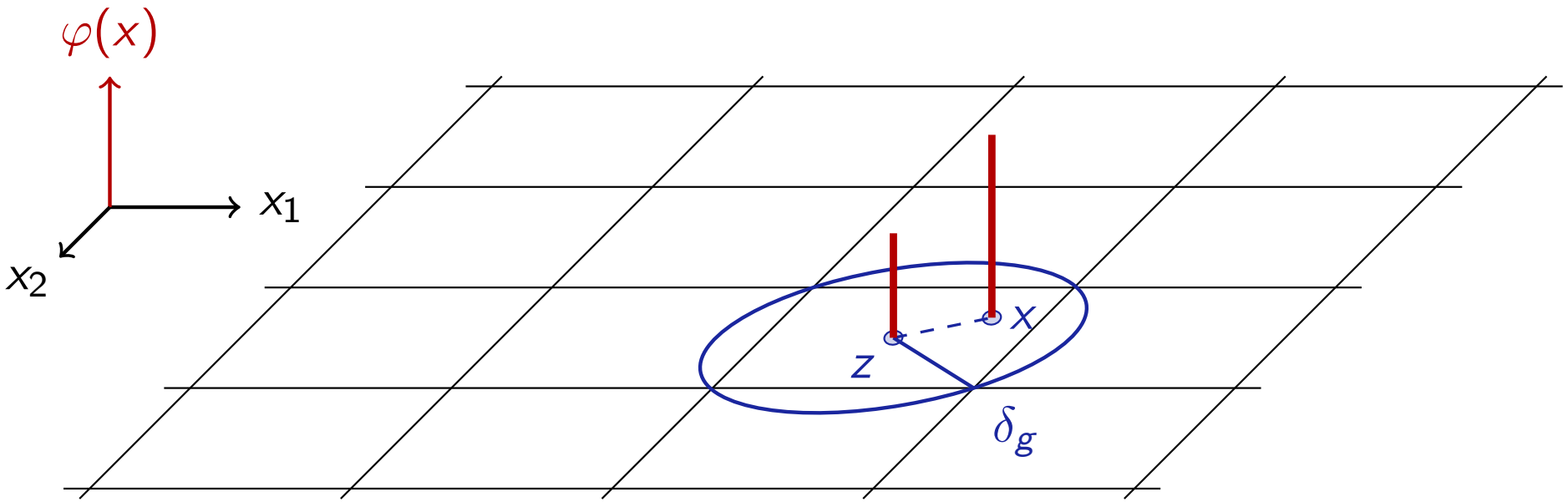

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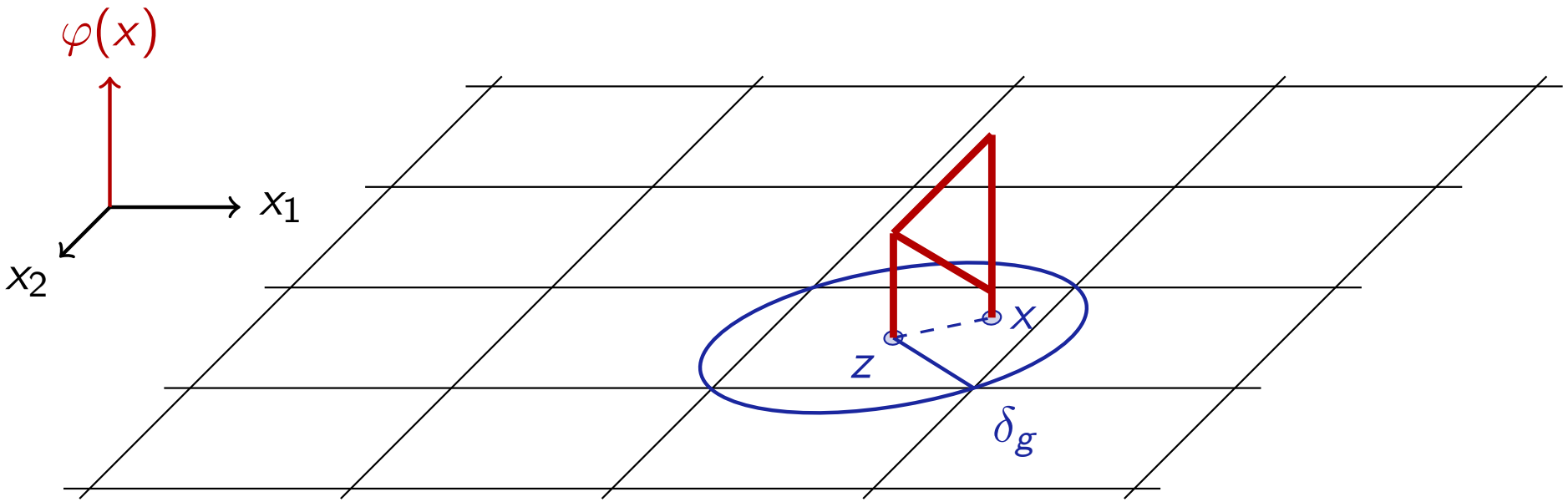

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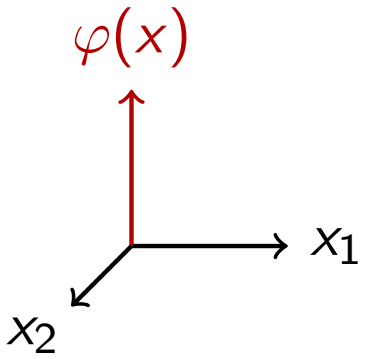

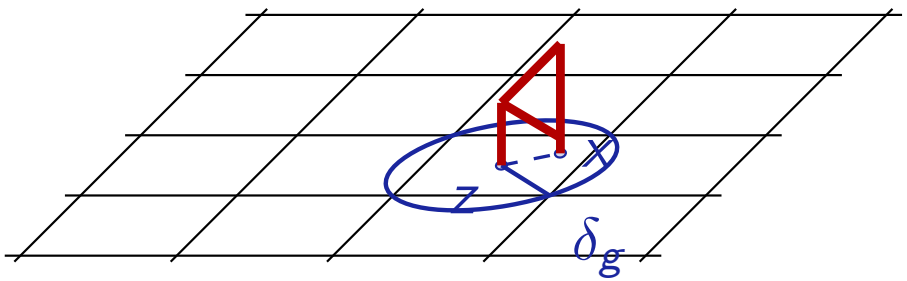

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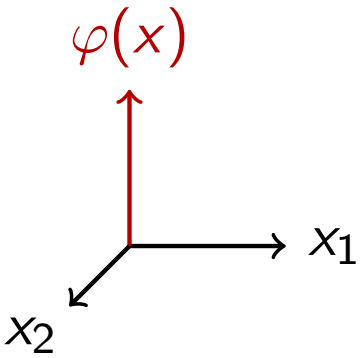

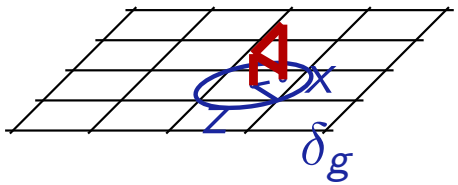

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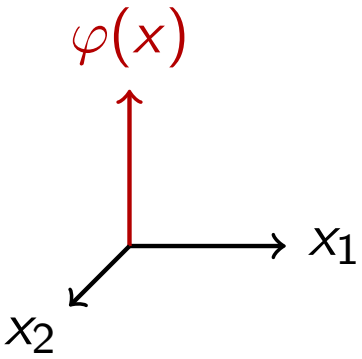

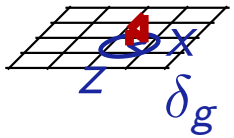

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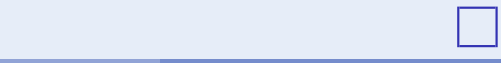

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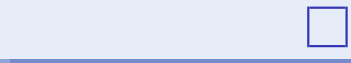
- Approximate by step functions $f_{\delta_c}(z)$ on $\pm\delta_g/2$ hypercube around z.



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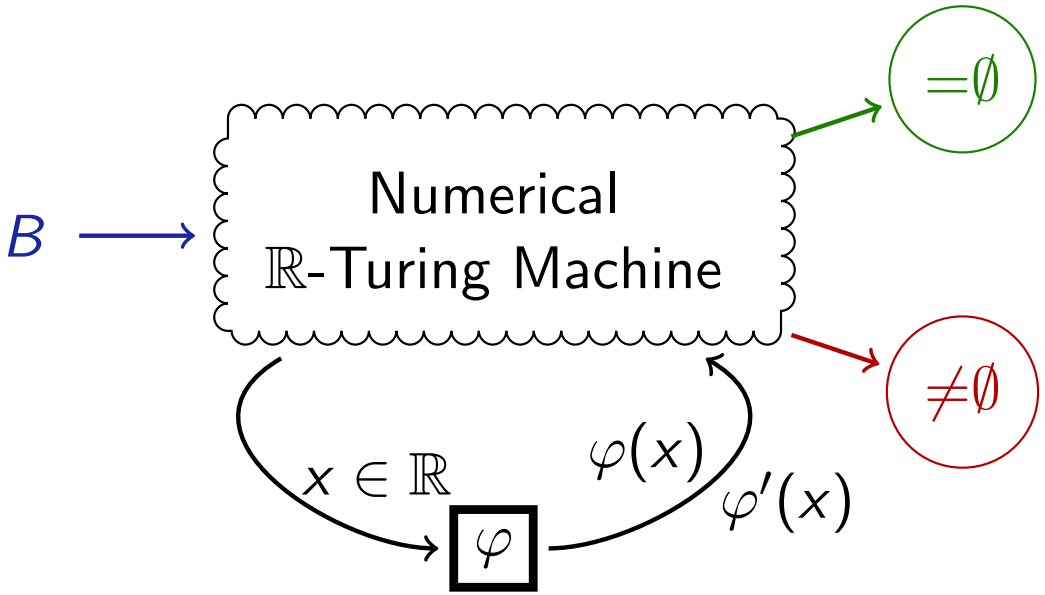
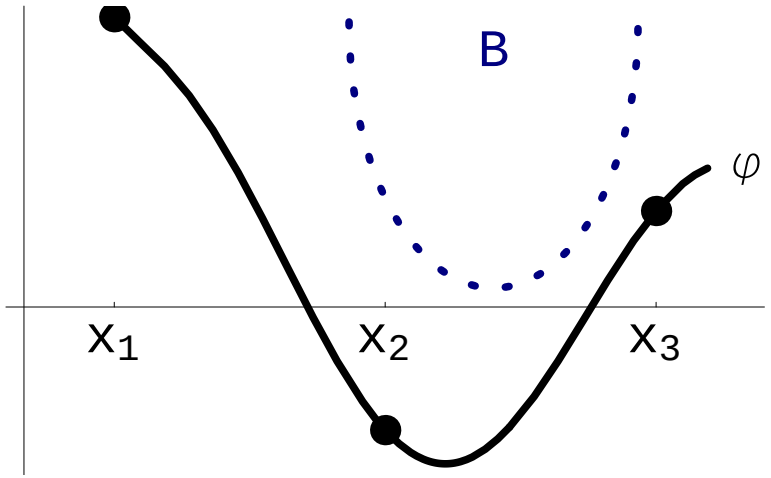
Only need to find the bound b . . .

Finding bounds is easier than verification?

Continuous Image Computation





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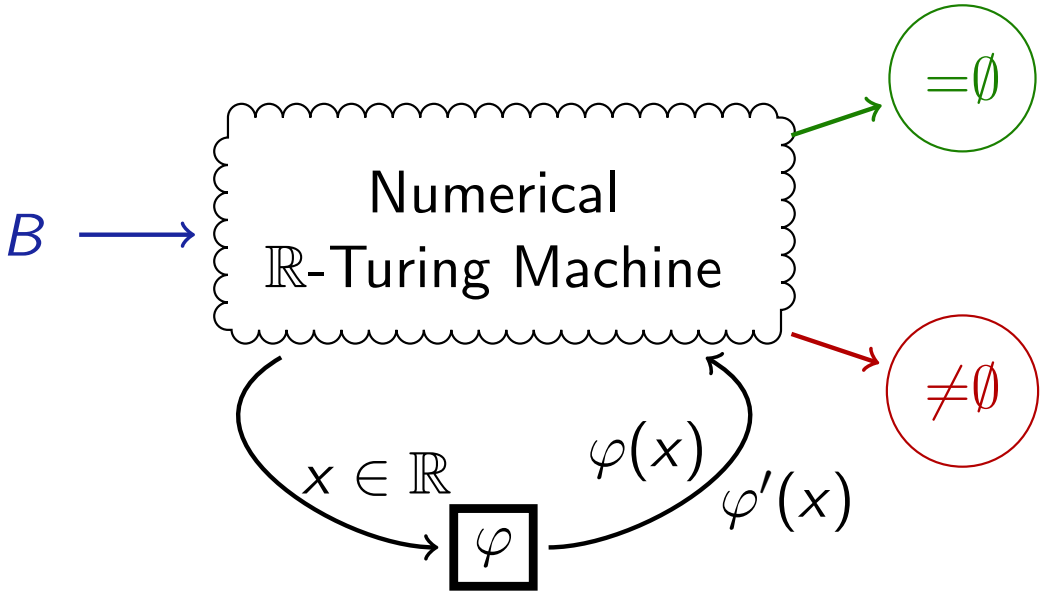

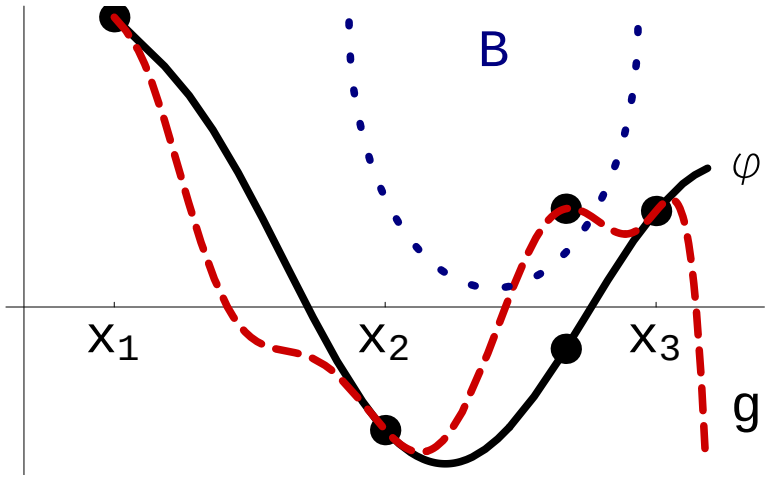

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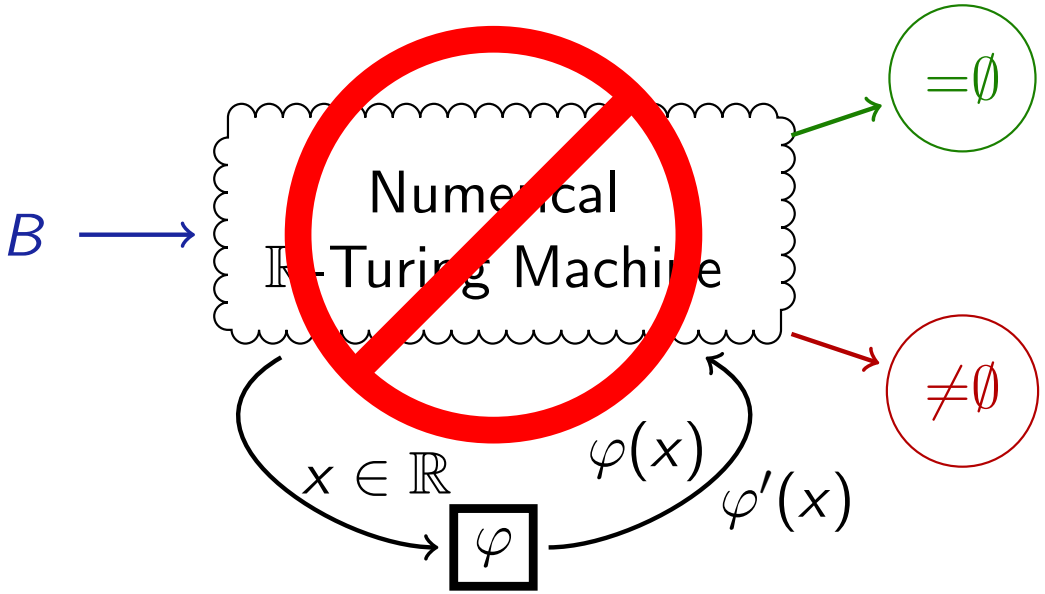

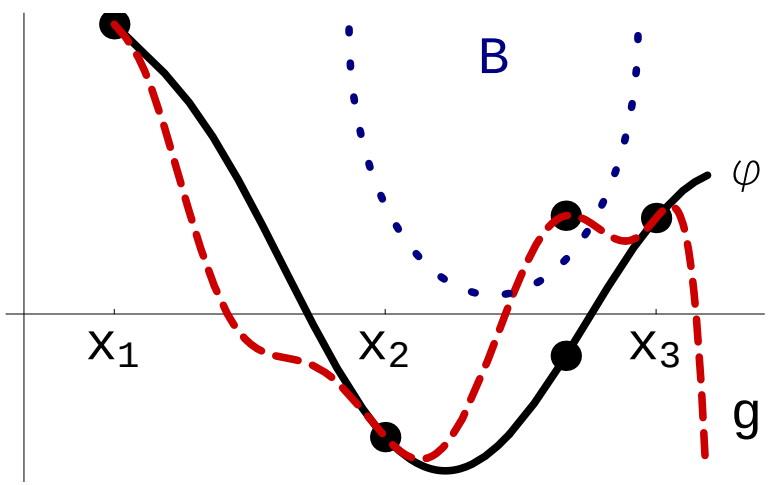

Proposition (Image computation undecidable for. . .)

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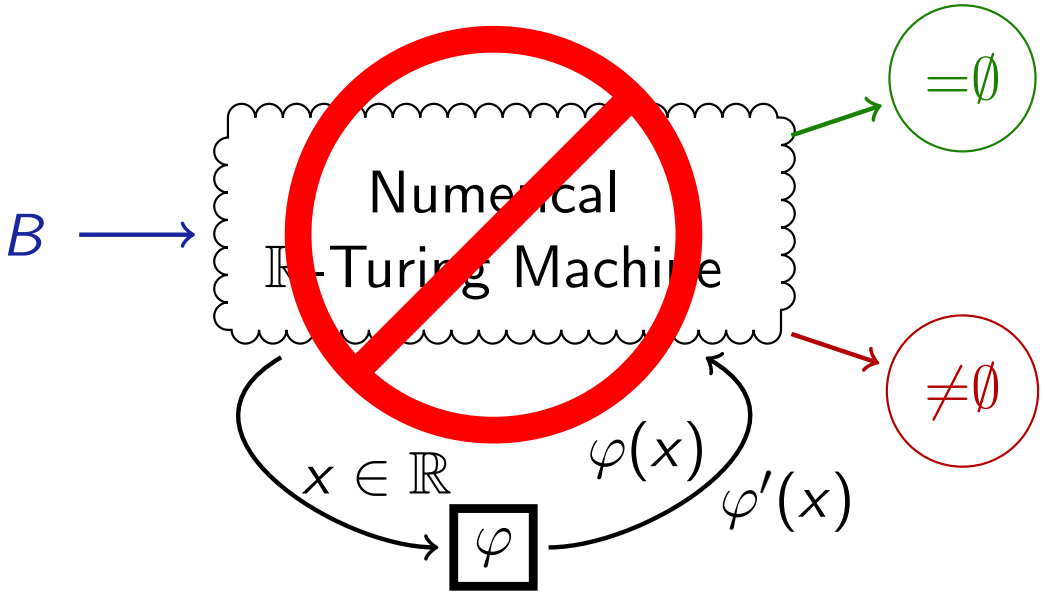

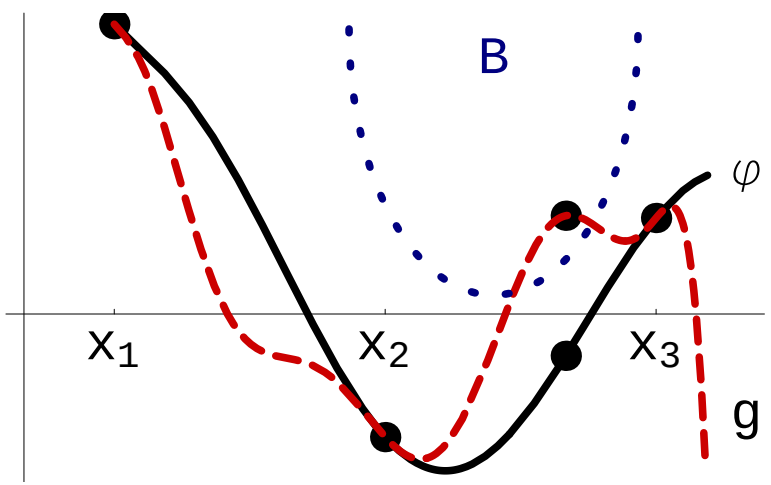

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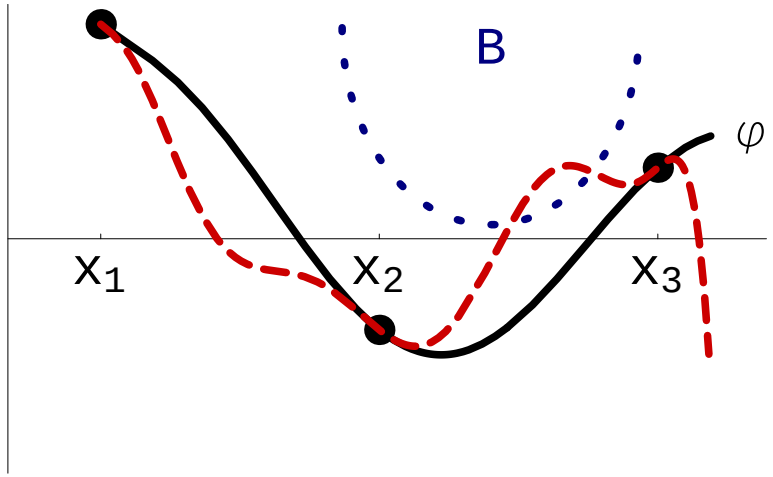
B
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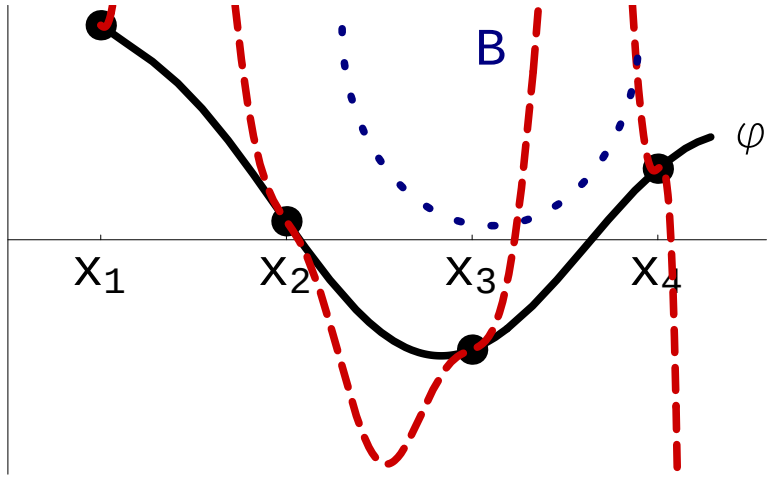
x2
x3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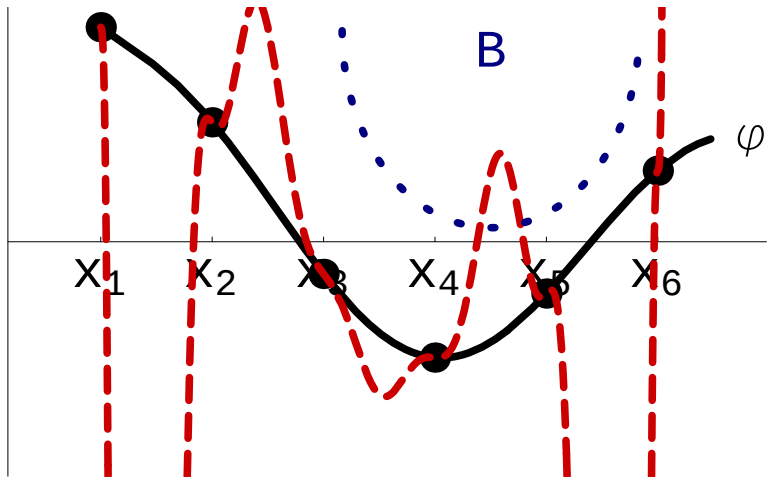
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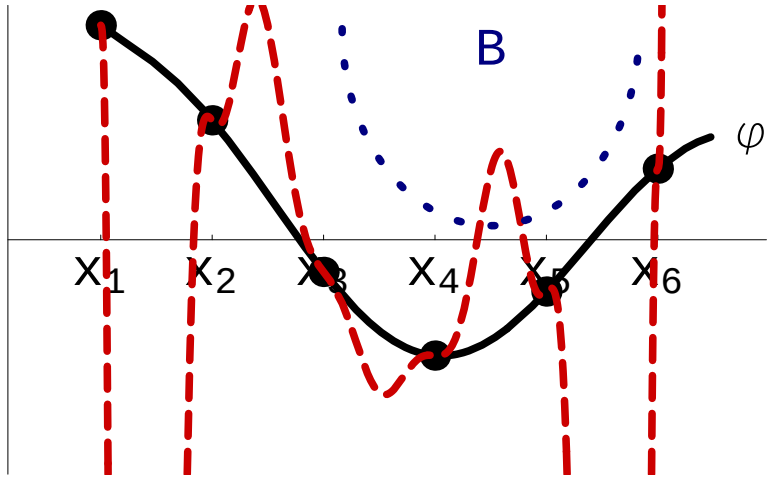
x5
x6

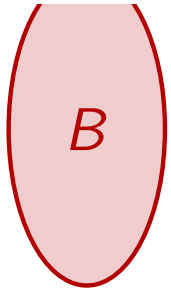


Proposition

- $P(\|\varphi'\|_\infty > b) \to 0$ *as* $b \to \infty$
- φ *evaluated on finite subset* $X = \{x_i\}$ *of open or compact* D

$\Rightarrow$ $P(\textit{decision correct}) \to 1$ *as* $\|d(\cdot, X)\|_\infty \to 0$

Probabilistic Model Checking: Proof Illustration



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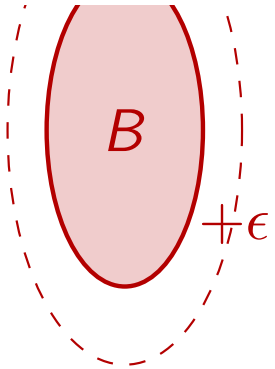

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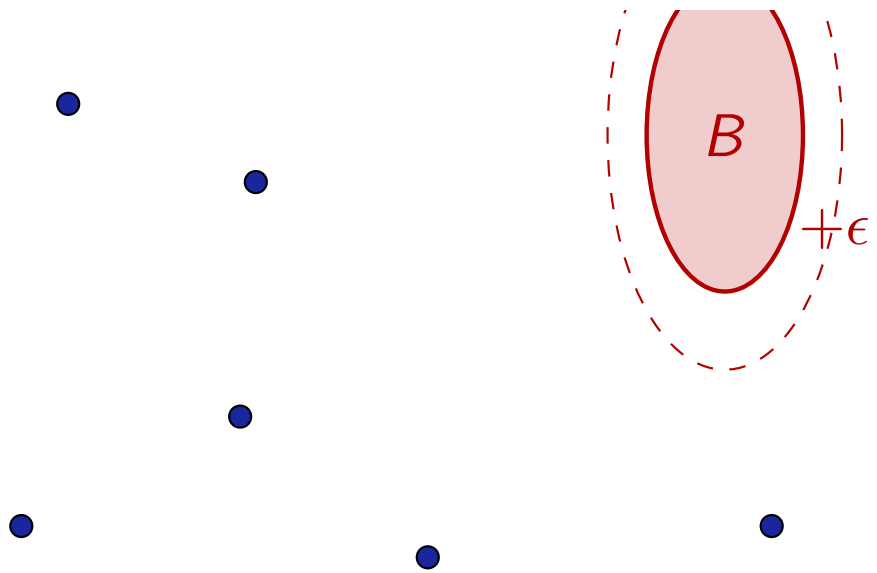

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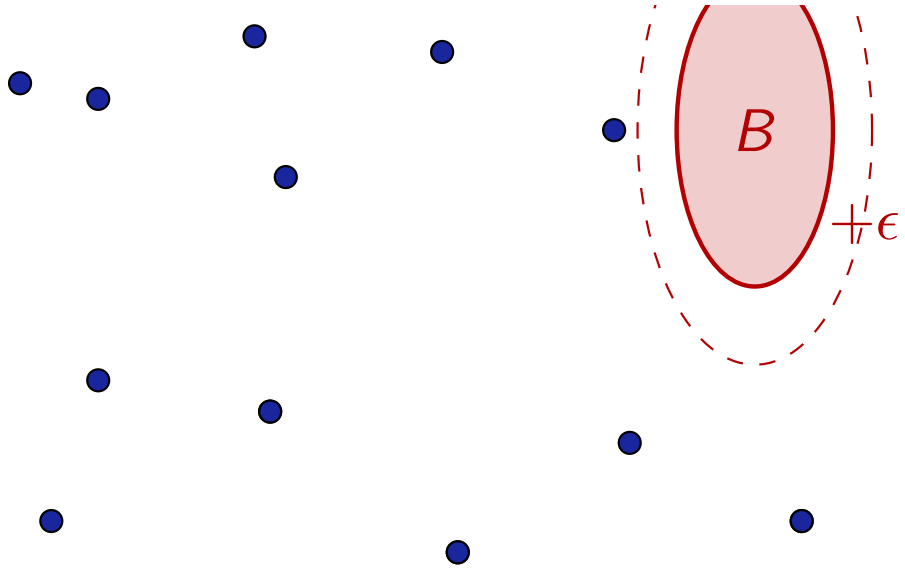

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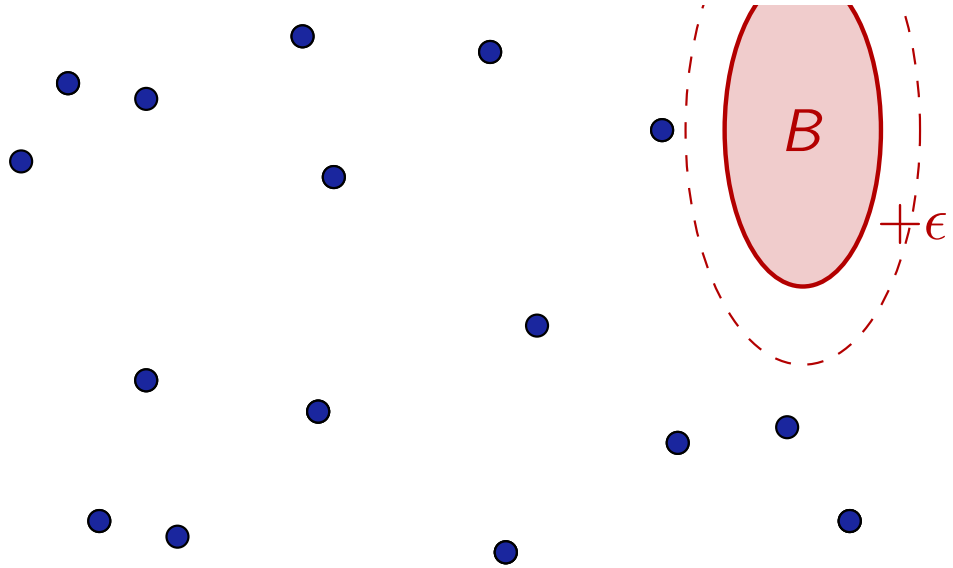

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Proof. (for problem with tolerance $\epsilon > 0$).

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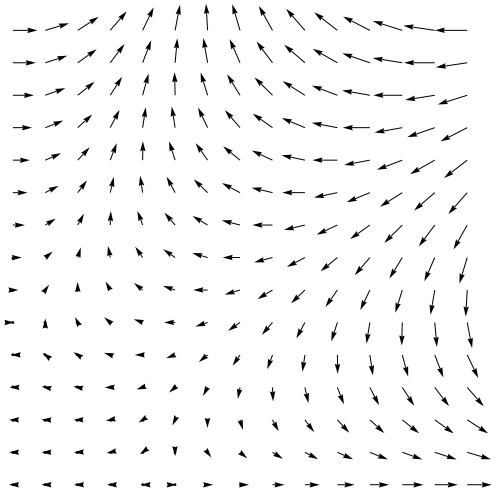
- Because $P(\|\varphi'\|_\infty \geq \frac{\epsilon}{\nu}) \to 0$ for $\nu \to 0$ by premise, as ϵ is a constant independent of ν and $\frac{\epsilon}{\nu} \to \infty$ as $\nu \to 0$.

Differential Flow Approximation



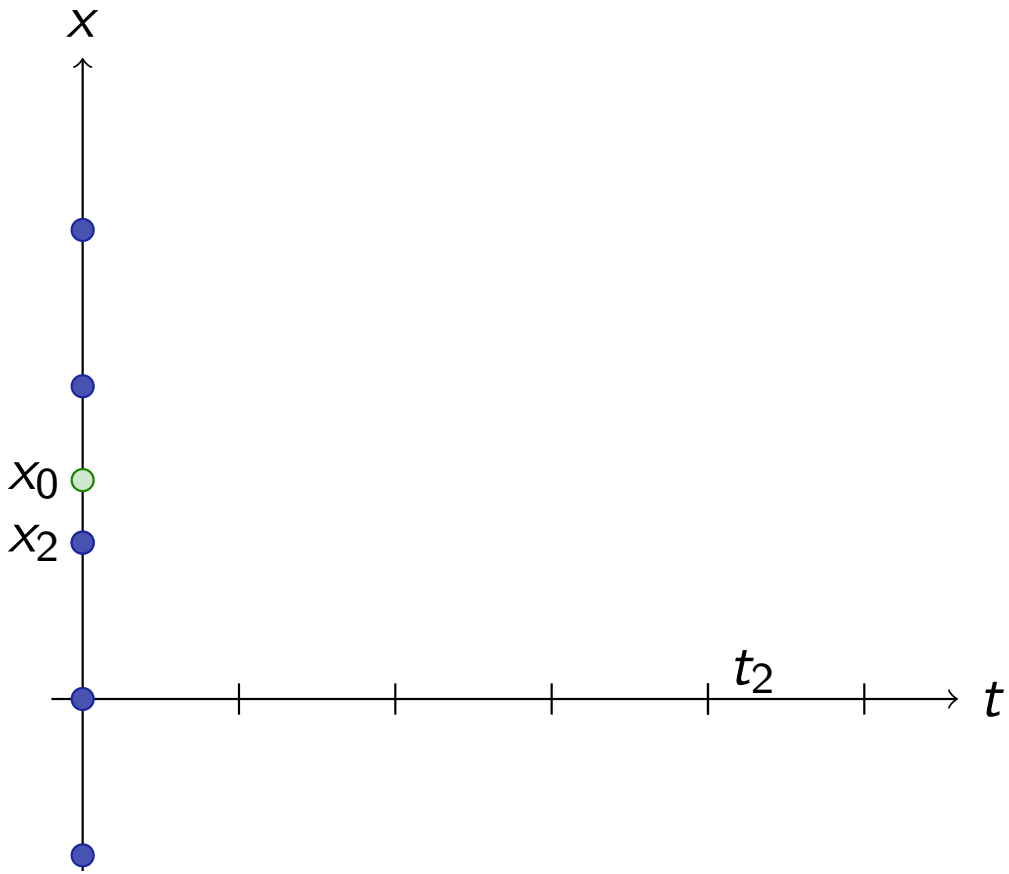
φ solves
$x'(t) = f(t, x)$

Proposition

- *Flow φ is solution of $x'(t) = f(t, x)$*
- $f \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$
- *ℓ-Lipschitz-continuous:* $\|f(t, x_1) - f(t, x_2)\| \leq \ell \|x_1 - x_2\|$

$\Rightarrow$ *Continuous image computation decidable*

Differential Flow Approximation: Proof Illustration



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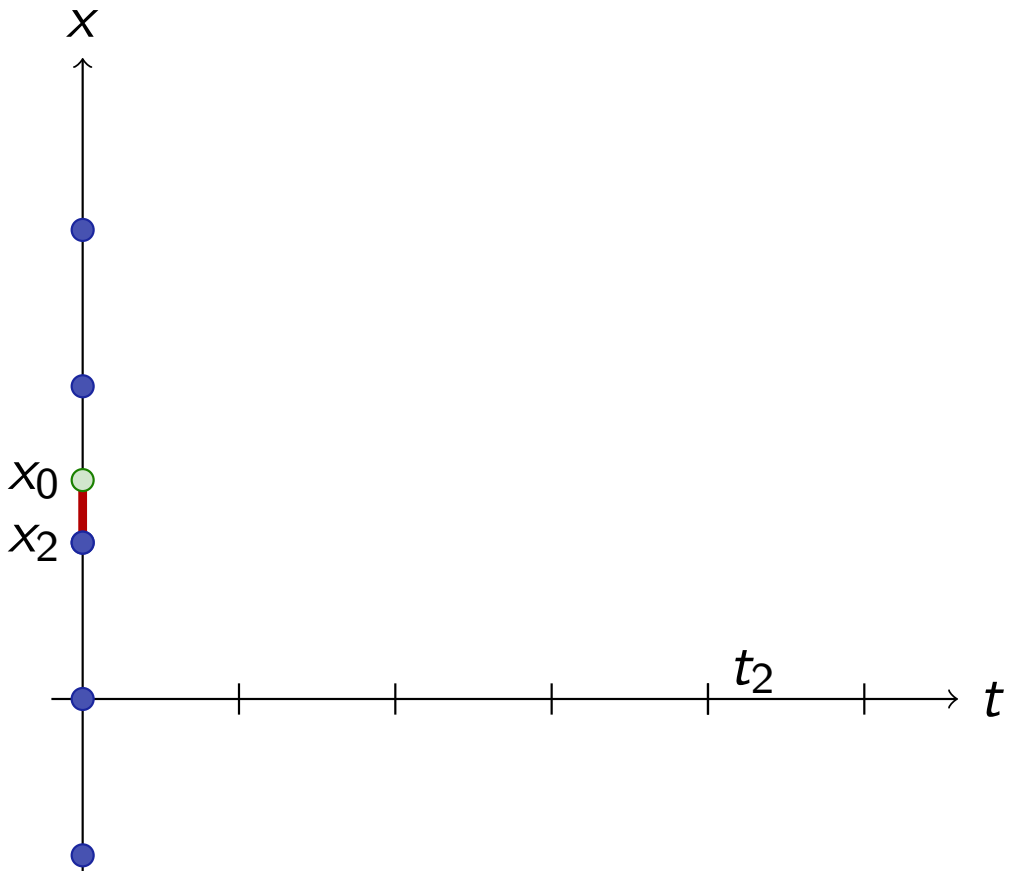

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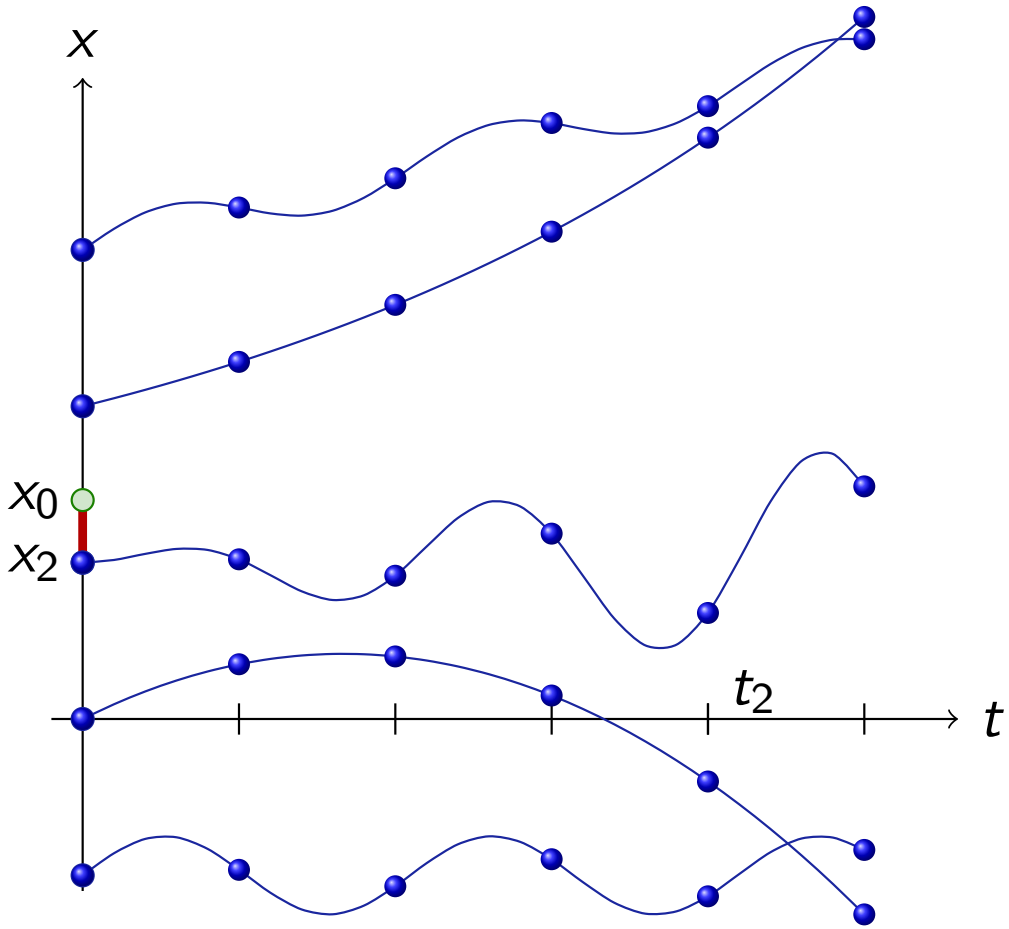

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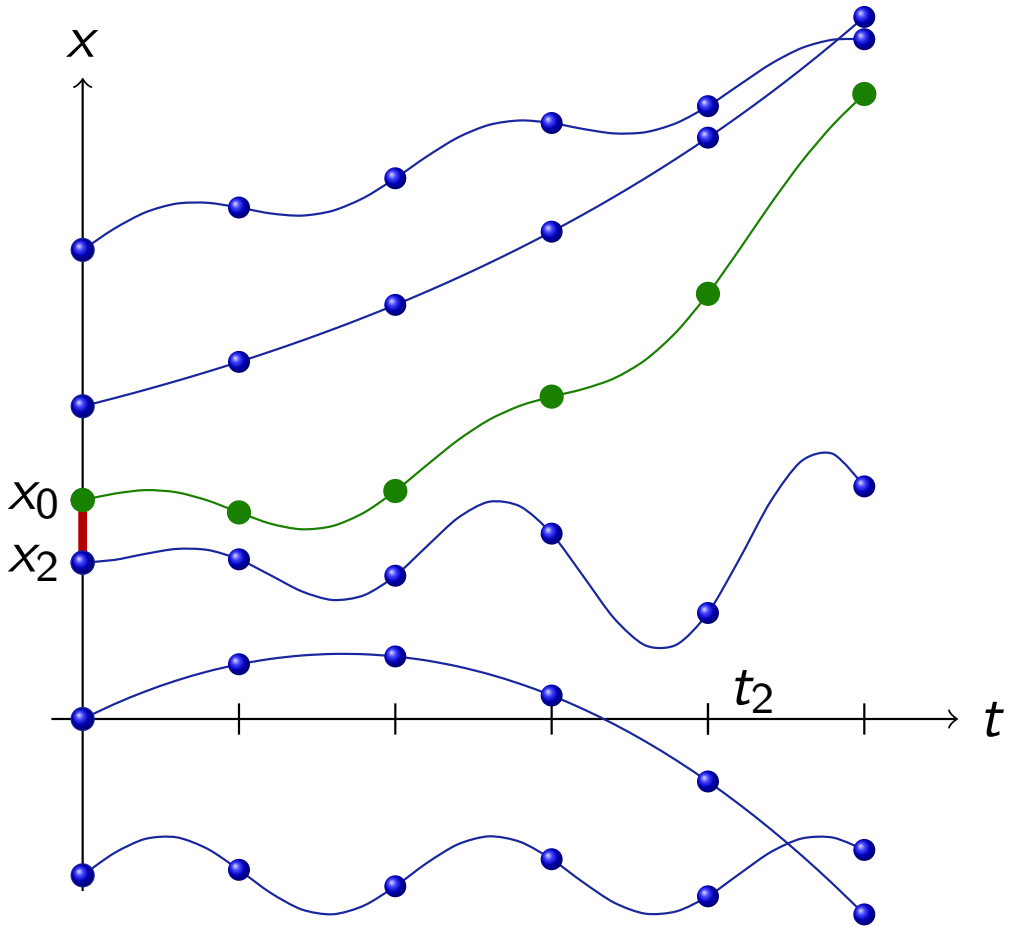

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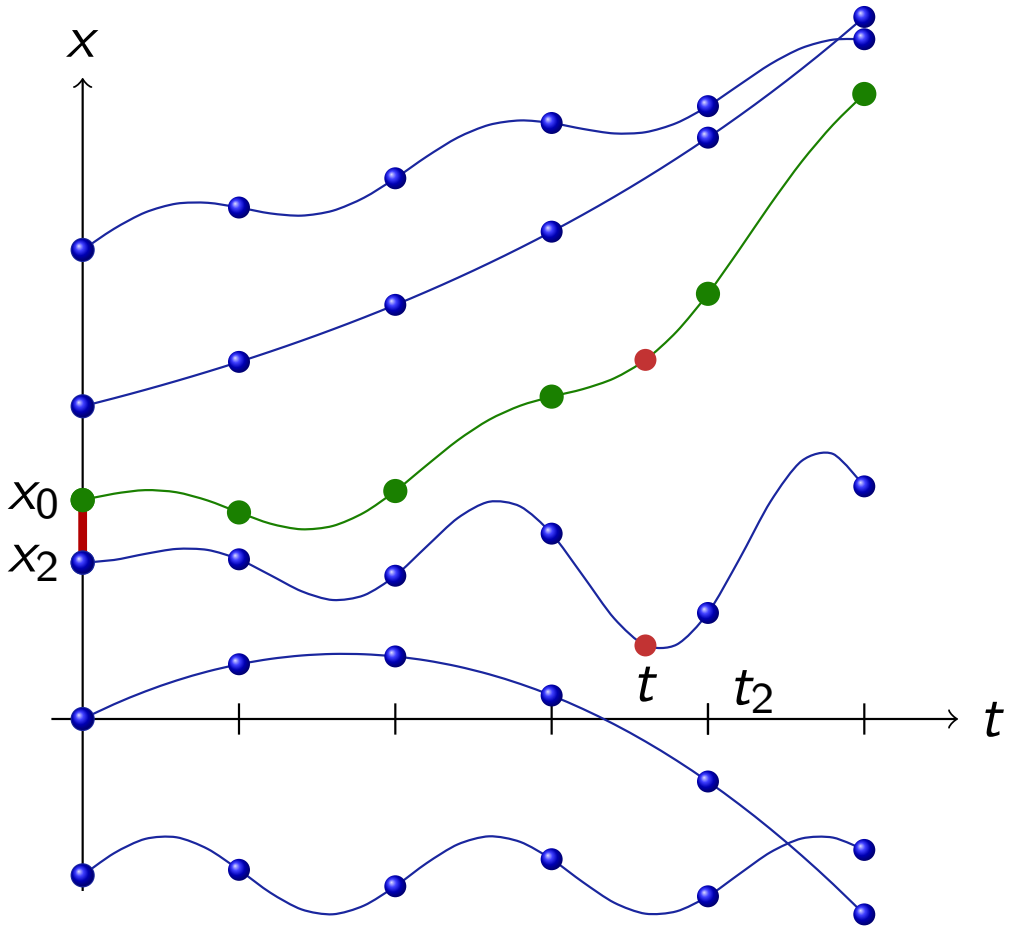

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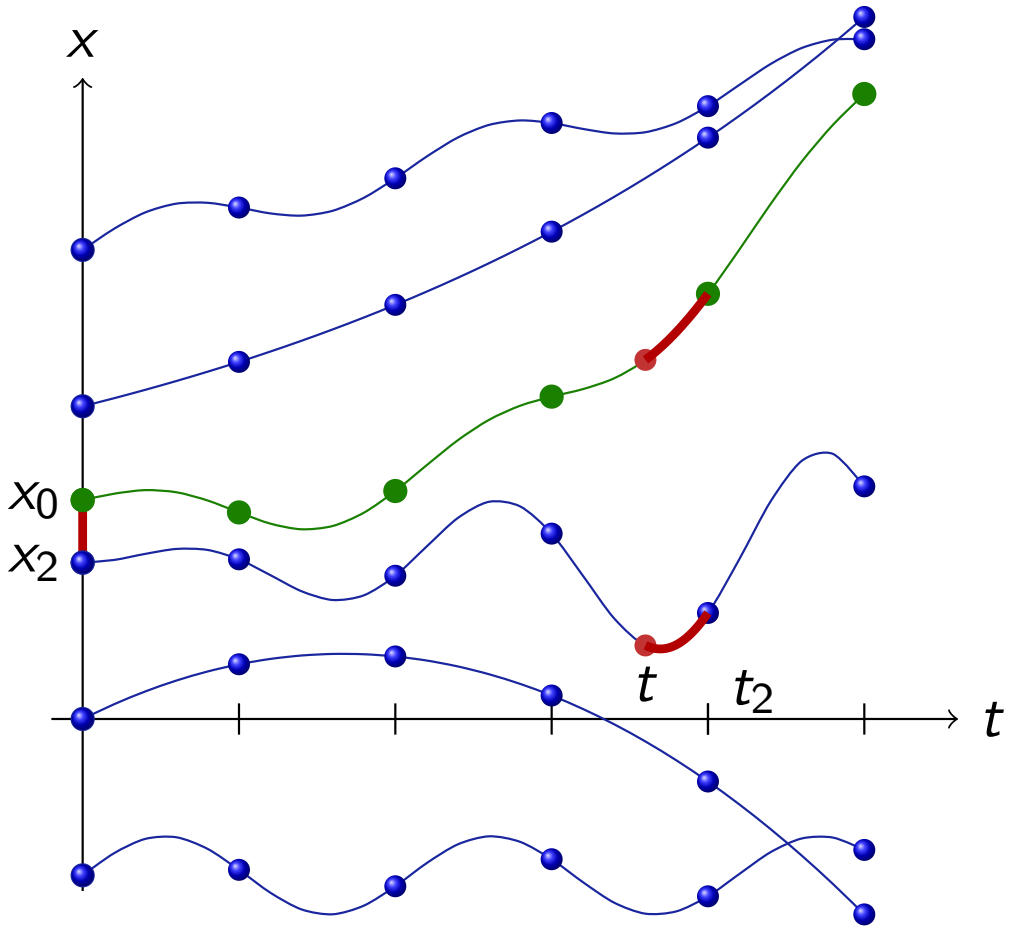

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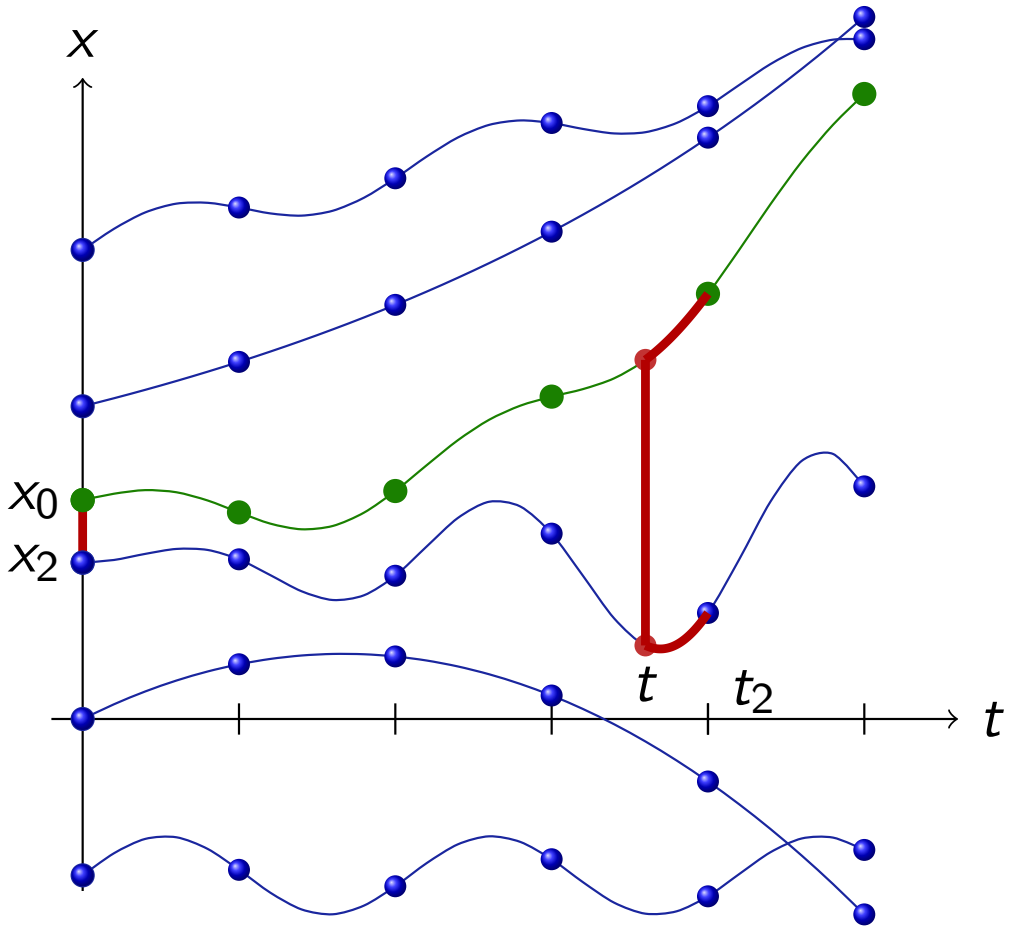

Differential Flow Approximation: Proof

Proof.

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by corollary of Picard-Lindelöf and MVT with $\xi \in (t, t_2)$.

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Approximation Errors

Exponential terms in approximation error computations are bad

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but tight!

Example

$$x' = \ell x$$

is ℓ-Lipschitz-continuous with unique global solution $\varphi(t; x_0) = x_0 e^{\ell(t-t_0)}$

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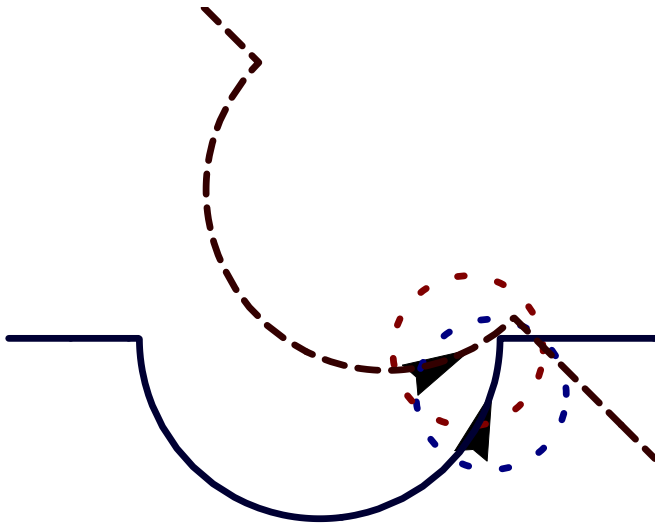
Outline

Experiments with Roundabout ATC

Counterexamples with distances ≈0.0016mi after 3 refinements

in absolute coords



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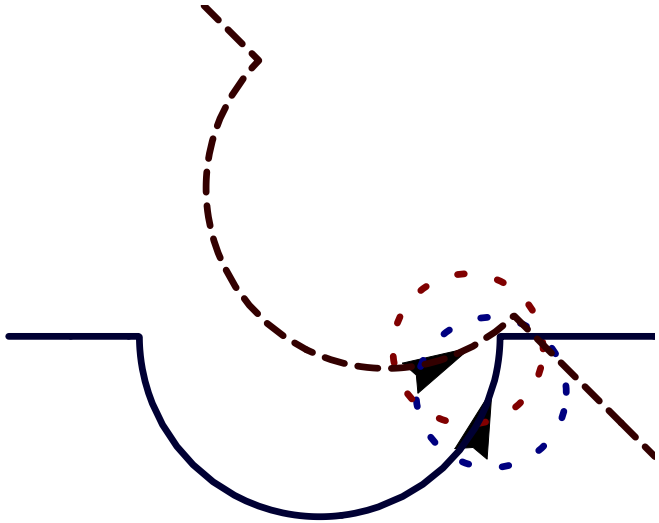
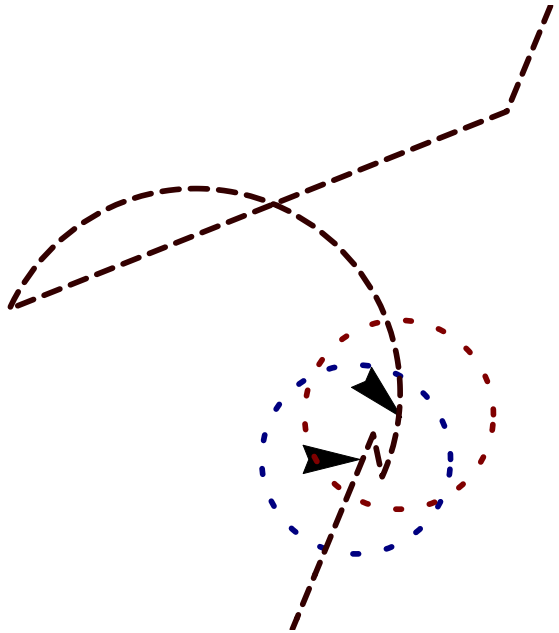


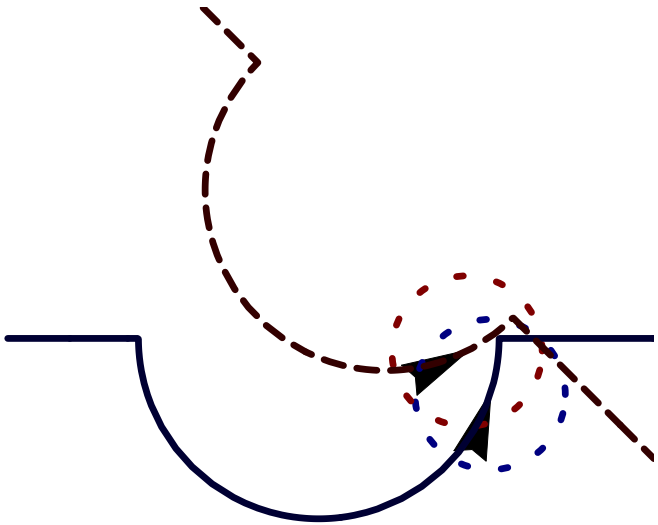
relative coords



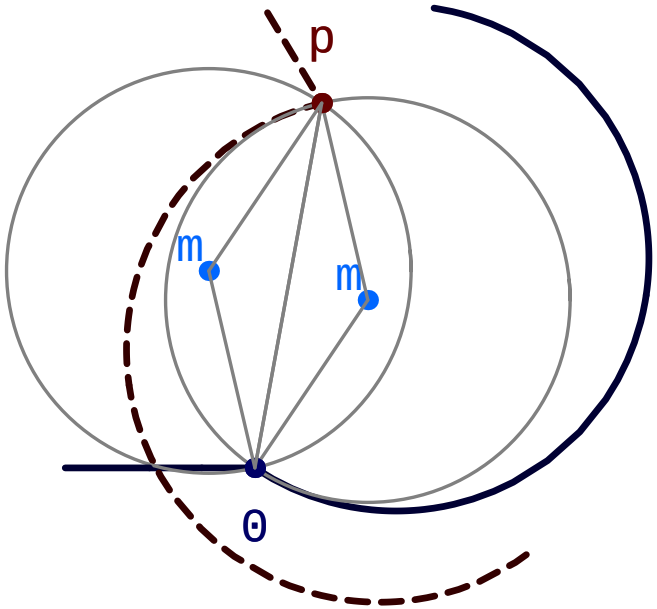
Experiments with Tangential Roundabout ATC

Solution: adaptively choose rotation using tangential construction

classical



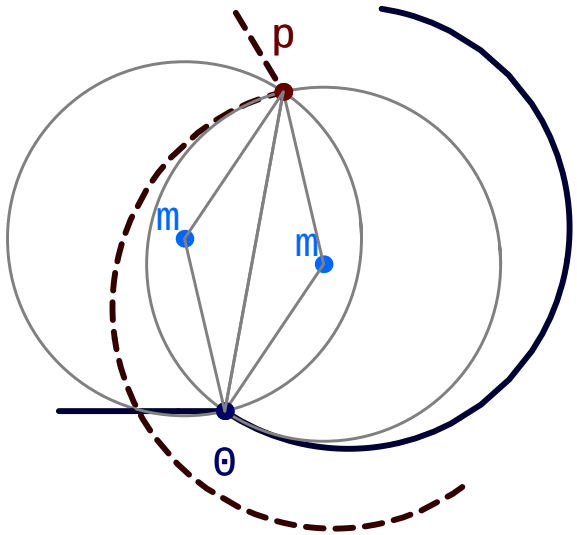
tangential



⊘ No more counterexamples found

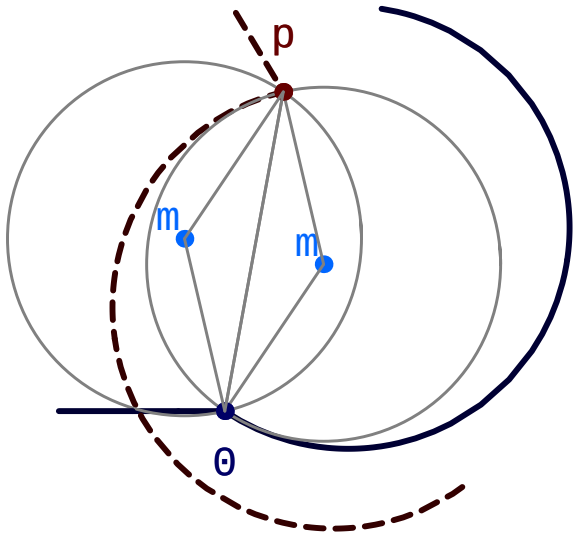
Experimental Results: Tangential Roundabout ATM

$$\alpha^2 = \|m - 0\|^2$$
$$\alpha^2 = \|m - p\|^2$$
$$\gamma_1 = \angle(m - 0)$$
$$\gamma_2 = \angle(m - p)$$



Experimental Results: Tangential Roundabout ATM

$$\begin{aligned}\alpha^2 &= \|m - 0\|^2 \\ \alpha^2 &= \|m - p\|^2 \\ \gamma_1 &= \angle(m - 0) \\ \gamma_2 &= \angle(m - p)\end{aligned}$$



Solutions for θ_j using any $k, \ell \in \{1, 2\}$:

$$\angle\left((-1)^{j+1}\frac{x^3 + xy^2 + (-1)^{j+k}\imath\sqrt{x^2(x^2+y^2)(4\alpha^2 - x^2 - y^2)}}{x(x - \imath y)}\right) + (-1)^{\ell}\frac{\pi}{2}$$

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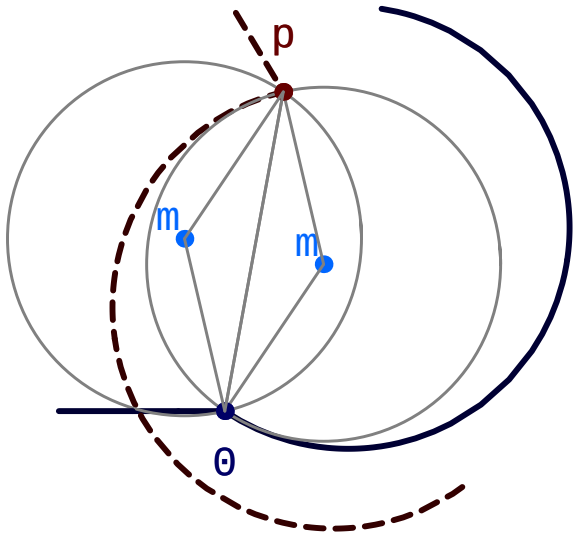

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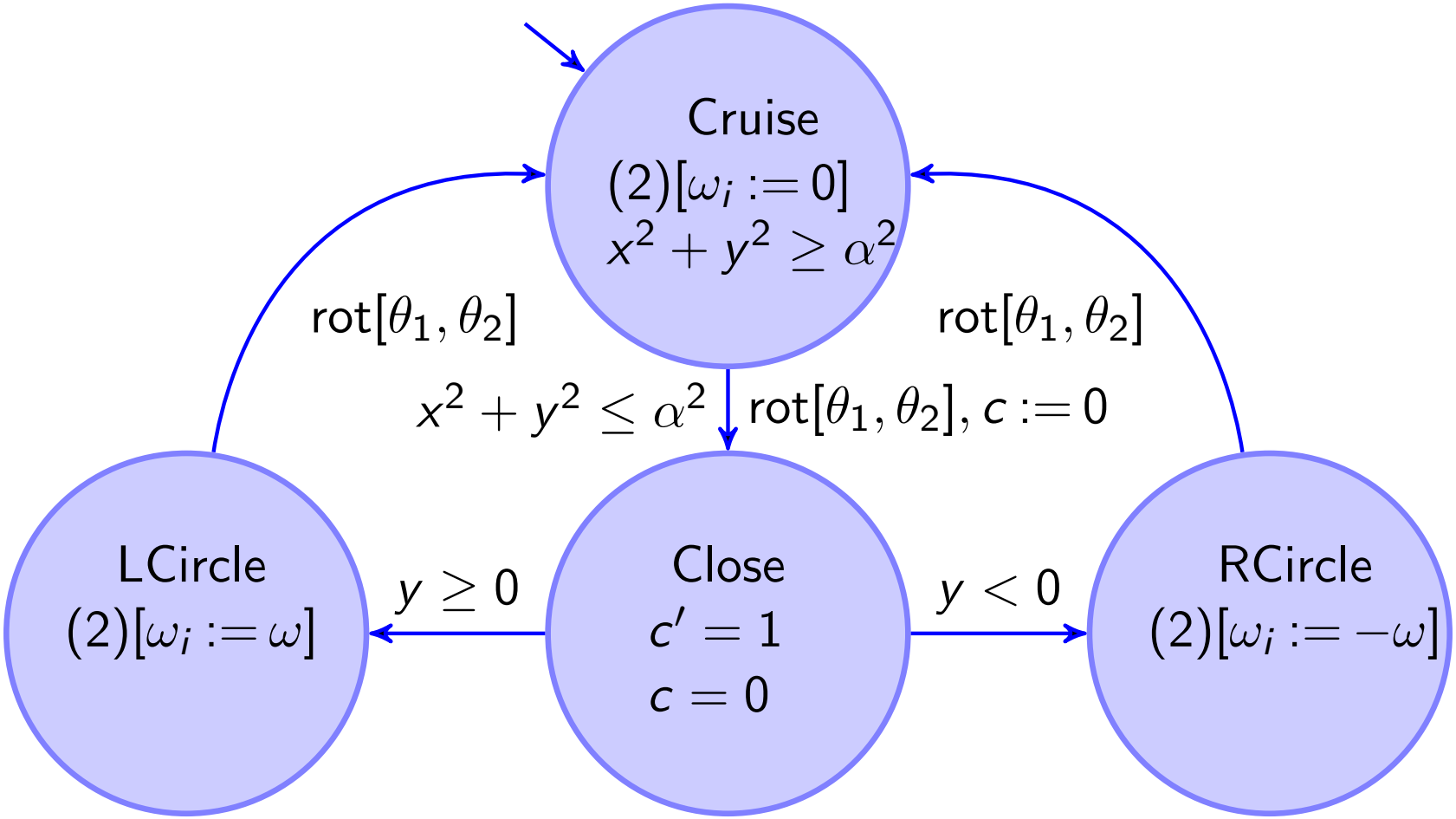
$$\min_{k,\ell} \max(|\theta_1 - 0|, |\theta_2 - \phi|)$$

Tangential Roundabout Maneuver Automaton



Return

Outline

Summary

- Image computation in hybrid systems model checking
  1. approx uniformly
  2. blur by uniform error
  3. check for B

flows	approx / image computation
continuous	uniform approx exists, but. . .
smooth	undecidable by evaluation
bounded by b	decidable
bound probabilities	probabilistically decidable
ODE ℓ-Lipschitz	decidable

- Combine numerical algorithms with symbolic analysis
- Roundabout maneuver unsafe
- Solution: adaptively choose rotations by tangential construction

Possible Extensions for Projects

- Extend tangential roundabout maneuver
  - Determine speed/thrust bounds
  - Position discrepancies caused by imprecise tracking
  - Verify liveness: aircraft finally on original route
  - Full curve dynamics
- Combine numerical algorithms with symbolic analysis ...
- Improved model checker
- Multivariate rational spline approximation